Rgularized hydrodynamic expansion from quantum fields

Outline

- Introduction: hydrodynamic in extreme conditions
- Method of moments for the Boltzmann equation
- Resummed moments expansion for the Wigner distribution
- Numerical results

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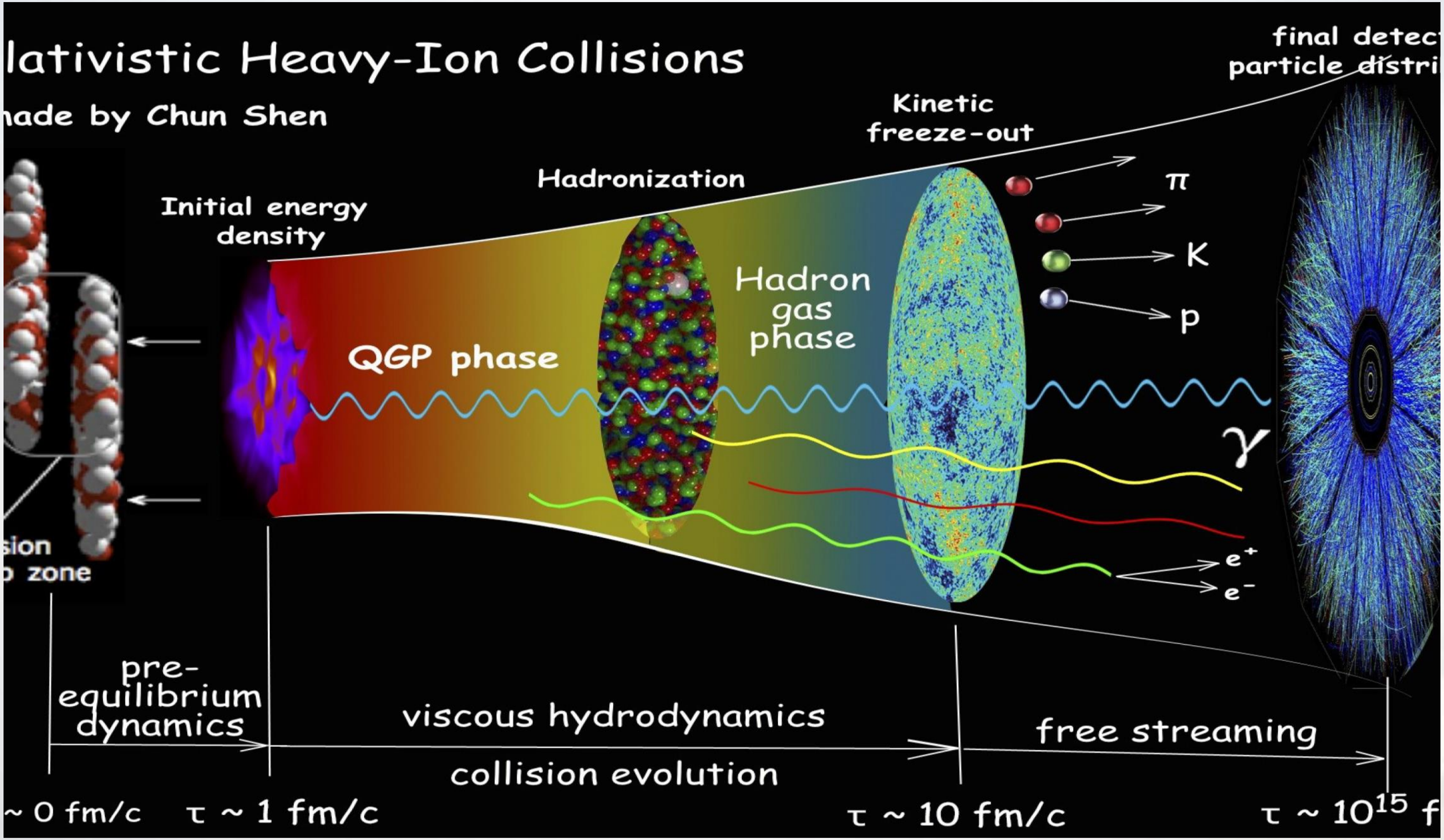
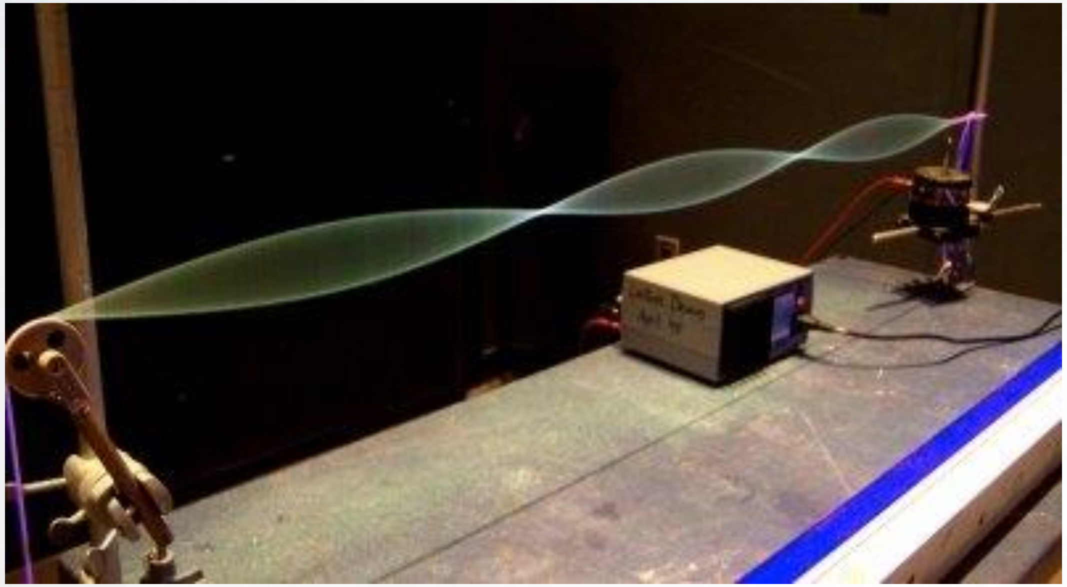
Leonardo Tinti Kraków, 26.11.2021

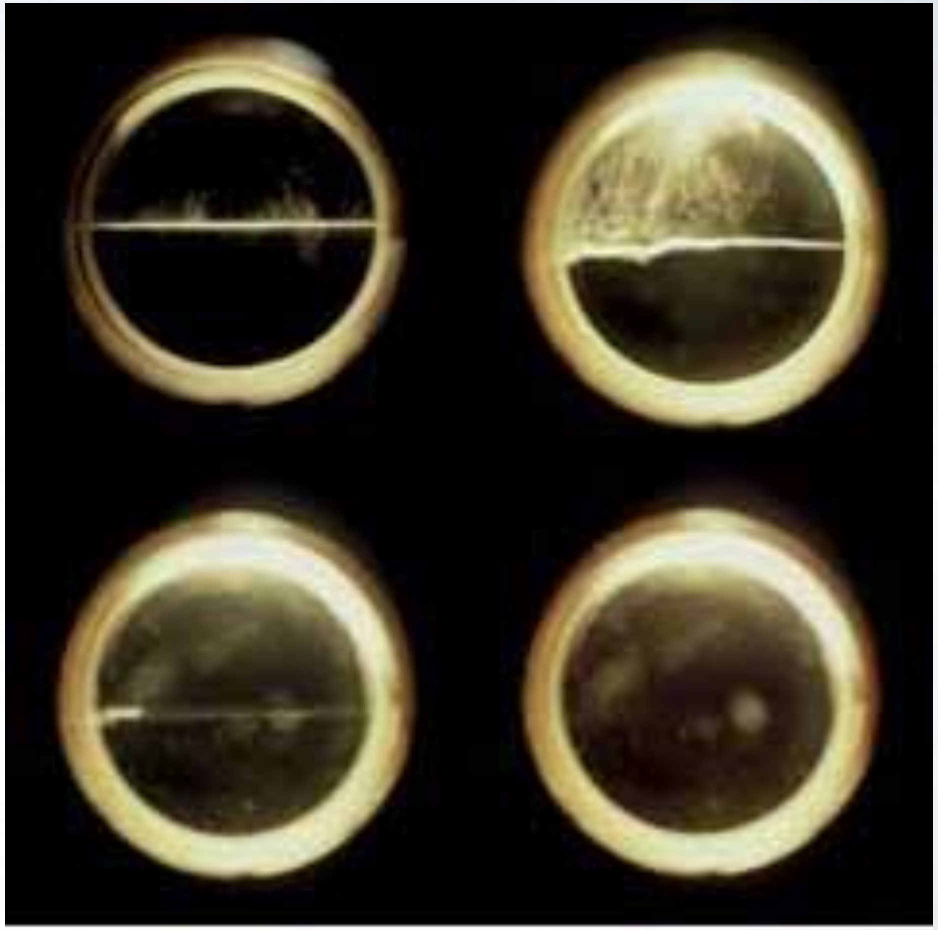
arXiv:1808.06436 arXiv:2003.09268

Standard picture of heavy ion collisions



Hydrodynamics

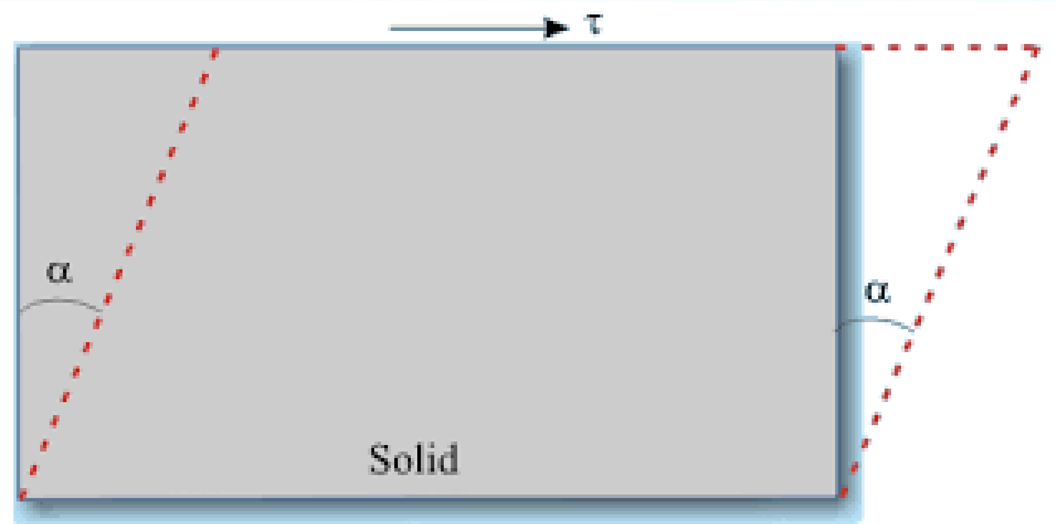




~~Hydrodynamics is the low-energy, long wave-length limit of a theory~~

~~Hydrodynamics require small gradients/deviations from equilibrium~~

Hydrodynamics



$$\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho\, \boldsymbol{v}) = 0$$

$$\rho \boldsymbol{a} = -\nabla_{\mathbf{x}} \mathrm{P}$$

generalizes to

$$\boxed{\rho \boldsymbol{a} = \nabla_x \cdot T}$$

the Cauchy Tensor T

definition of a fluid:

$$\boxed{T_{ij}\Big|_{eq} = -\mathrm{P}\, \delta_{ij}}$$

For an incompressible fluid $\nabla_x \cdot \boldsymbol{v} = \mathbf{0}$

$$T_{ij} \simeq -\mathrm{P}\, \delta_{ij} + \eta \left(\partial_i v_j + \partial_j v_i \right) + \cdots$$

Relativistic hydrodynamics

$$\boldsymbol{v} \to u = \begin{pmatrix} \gamma \\ \gamma\boldsymbol{v}/c \end{pmatrix}$$
$$\rho \to \mathcal{E}$$

relativistic degrees of freedom

projector: $$\Delta^{\mu\nu} = u^\mu u^\nu - g^{\mu\nu}$$

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu - \mathrm{P}\Delta^{\mu\nu} + \cdots$$

$$\partial_\mu \mathrm{T}^{\mu\nu} = 0$$

local four-momentum conservation

which implies

$$\begin{cases} 0 = u_\nu \partial_\mu T^{\mu\nu} \xrightarrow{c\to\infty} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho\, \boldsymbol{v}) \\ 0 = \Delta_{i\nu} \partial_\mu T^{\mu\nu} \xrightarrow{c\to\infty} (\rho \boldsymbol{a} + \nabla_{\mathbf{x}} \mathrm{P}) \Big|_i + \cdots \end{cases}$$

Relativistic hydrodynamics

$$\left.\begin{aligned} \partial_\mu \hat{T}^{\mu\nu} &= 0 \\ T^{\mu\nu} &= tr(\hat{\rho}\,\hat{T}^{\mu\nu}) \end{aligned}\right\} \rightarrow \partial_\mu T^{\mu\nu} = 0$$

Hydro

$$T^{\mu\nu} = \mathcal{E}\, u^\mu\, u^\nu - \mathcal{P}\Delta^{\mu\nu} + \delta T^{\mu\nu}$$

From quantum field theory, but at least ten degrees of freedom and only four equations

Gradient expansion

- Requires small gradients
- Unstable (even in the non-relativistic limit)
- Not converging

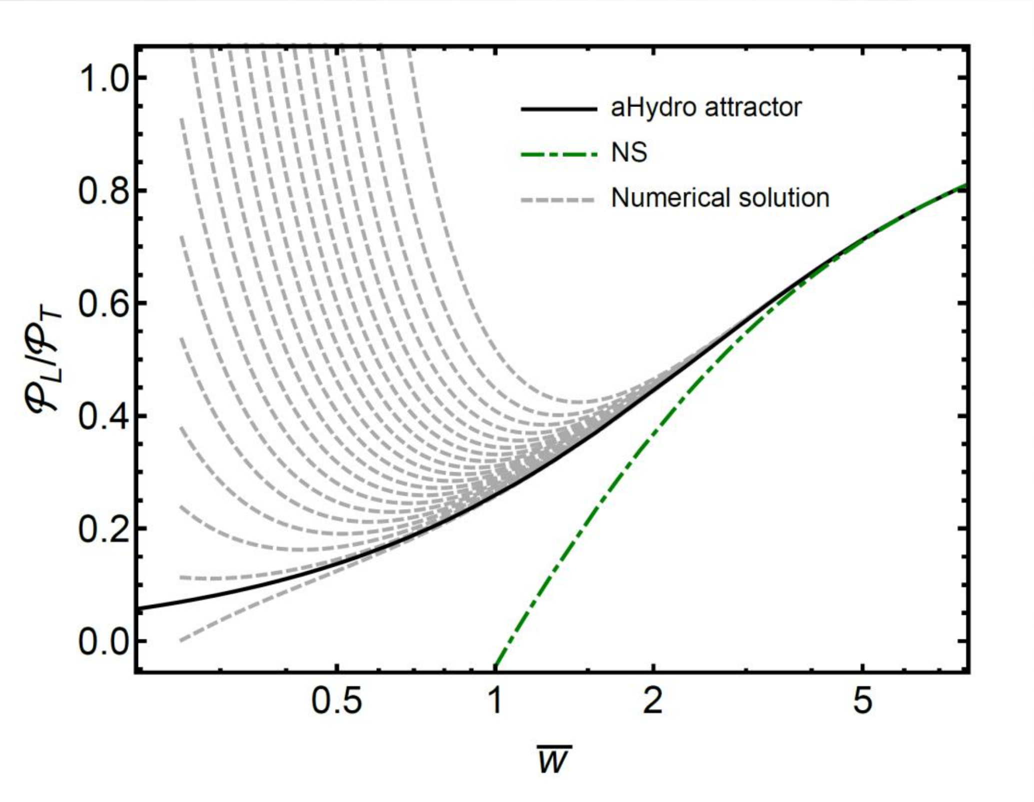
A Buchel, M P Heller, J Noronha, arXiv:1603.05344

G Denicol, J Noronha, arXiv:1608.07869

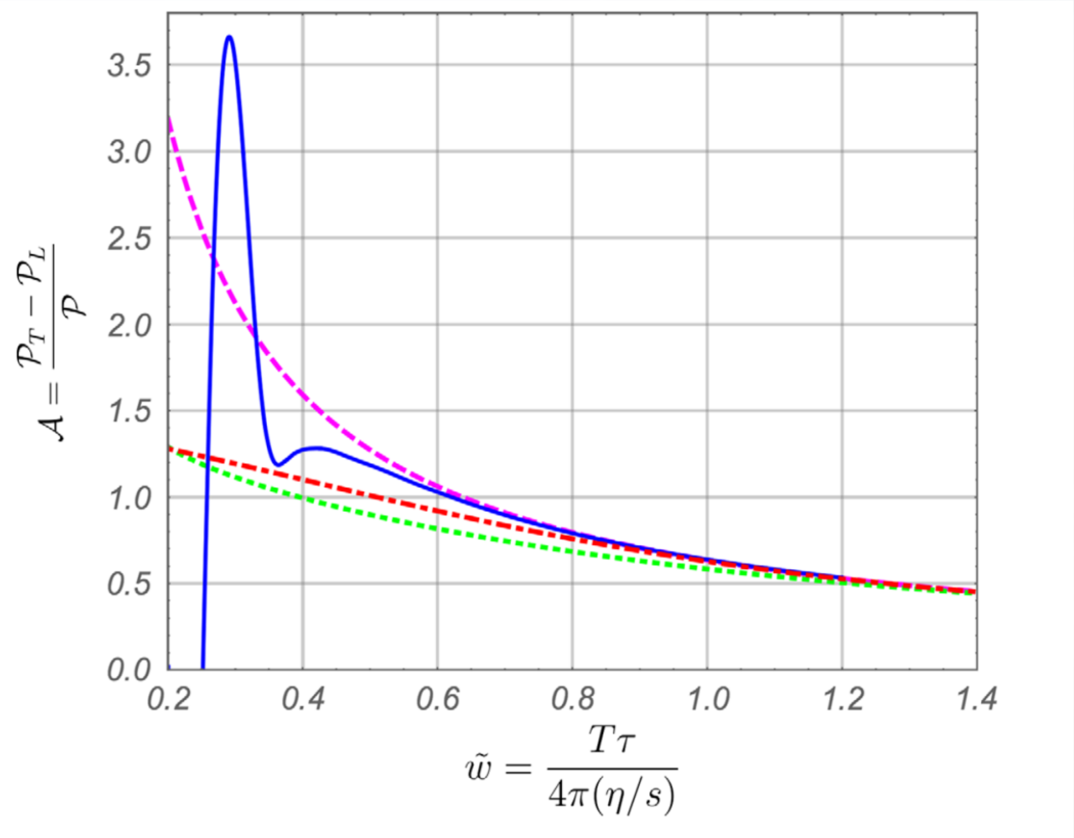
$$\delta T^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \cdots$$

transport coefficients times gradients

If the gradient expansion diverges, can hydrodynamics mak sense?



M Strickland, J Noronha, G Denicol
arXiv:1709.06644



M P. Heller, A Kurkela, M Spalinski, V Svensson
arXiv:1609.04803

Attractor behavior! …but A Behtash, C N Cruz-Camacho, M Martinez arXiv:1711.01745

From the kinetic theory ("almost alternative")

Relativistic Boltzmann equation

$$p \cdot \partial f = -\mathcal{C}[f]$$

$$\Rightarrow \underbrace{\int_{\boldsymbol{p}} p^{\nu} p \cdot \partial f}_{\partial_{\mu} T^{\mu\nu}} = -\int_{\boldsymbol{p}} p^{\nu} C = 0$$

$$u \cdot \partial f = \dot{f} = -\frac{p \cdot \nabla f}{(p \cdot u)} - \frac{C[f]}{(p \cdot u)}$$

extra needed equations

$$\dot{T}^{\mu\nu} = \int_{\boldsymbol{p}} p^{\mu} p^{\nu} \dot{f}$$

covariant momentum integral

$$\int_{\mathbf{p}} = \int d^4 p \, 2\Theta(p_0) \delta(p^2 - m^2)$$

G Denicol, J.Phys. G41 (2014) no.12, 124004

From the kinetic theory

$$p \cdot \partial f = -\mathcal{C}[f]$$

convenient basis $$\boxed{\mathcal{F}_r^{\mu_1 \cdots \mu_s} = \int_{\mathbf{p}} (p \cdot u)^r \, p^{\mu_1} \cdots p^{\mu_s} f}$$

$$\dot{\mathcal{F}}_r^{\mu_1 \cdots \mu_s} + C_{r-1}^{\mu_1 \cdots \mu_s} = r \, \dot{u}_\alpha \, \mathcal{F}_{r-1}^{\alpha \mu_1 \cdots \mu_s} - \nabla_\alpha \mathcal{F}_{r-1}^{\alpha \mu_1 \cdots \mu_s} + (r-1) \nabla_\alpha u_\beta \, \mathcal{F}_{r-2}^{\alpha \beta \mu_1 \cdots \mu_s}$$

special case: $T^{\mu\nu}$ (r=0,s=2) $$\dot{T}^{\mu\nu} + C_{-1}^{\mu\nu} = -\nabla_\alpha \mathcal{F}_{-1}^{\alpha\mu\nu} - \nabla_\alpha u_\beta \, \mathcal{F}_{-2}^{\alpha\beta\mu\nu}$$

in particular $$u_\nu \dot{T}^{\mu\nu} + C_0^\nu = -u_\nu \left(\nabla_\alpha \mathcal{F}_{-1}^{\alpha\mu\nu} + \nabla_\alpha u_\beta \mathcal{F}_{-2}^{\alpha\beta\mu\nu} \right) \Rightarrow \partial_\mu T^{\mu\nu} = 0$$

From the kinetic theory

$$\mathcal{O}^{\langle\mu_1\rangle\cdots\langle\mu_l\rangle} = \Delta^{\mu_1}_{\alpha_1}\cdots\Delta^{\mu_l}_{\alpha_l}\mathcal{O}^{\alpha_1\cdots\alpha_l}$$

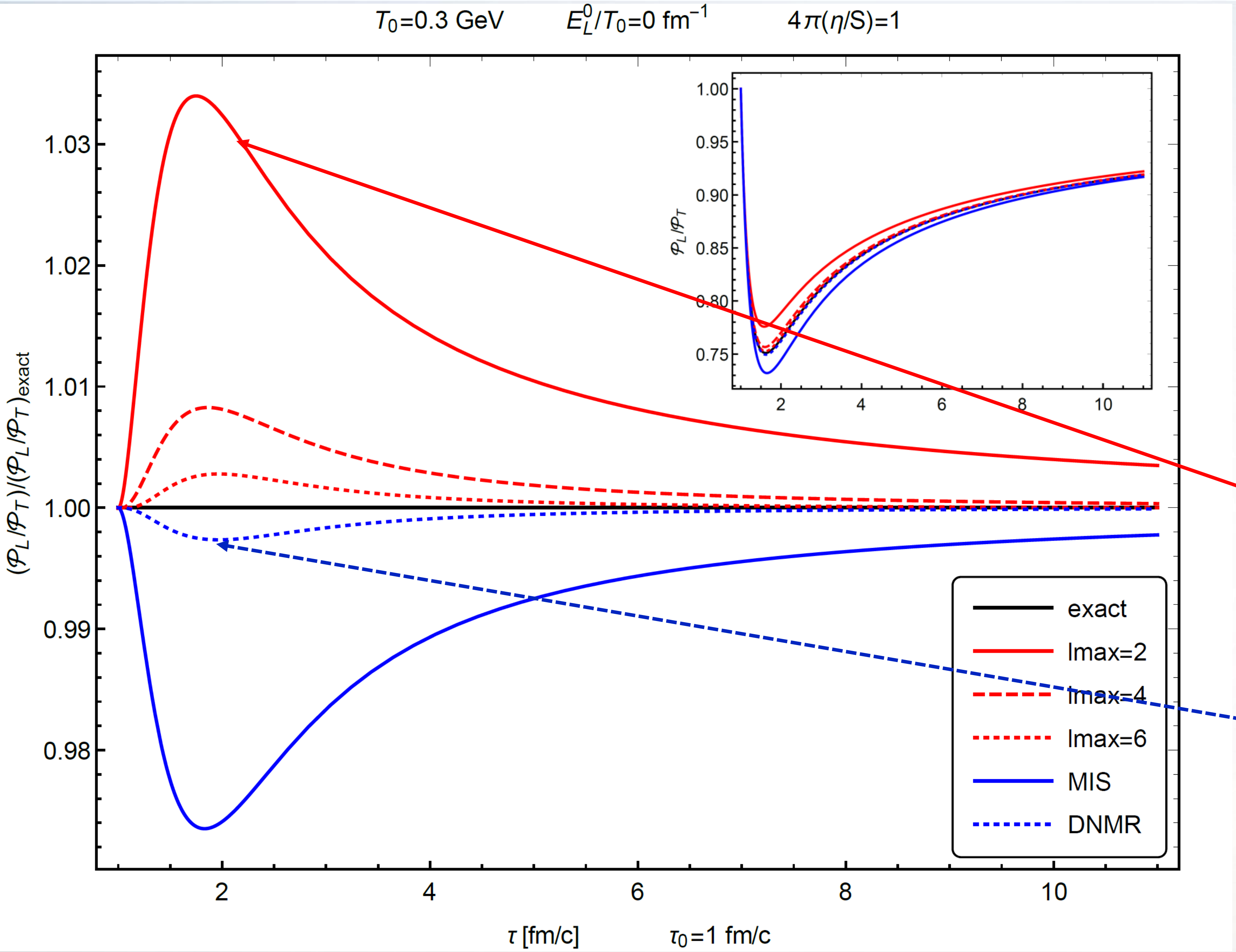
even more convenient basis

$$\mathfrak{f}_r^{\mu_1\cdots\mu_l} = \int_{\boldsymbol{p}} (p\cdot u)^r\, p^{\langle\mu_1\rangle\cdots\langle\mu_l\rangle} f$$

$$\partial_\mu u_\nu = u_\mu \dot{u}_\nu + \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3}\theta\Delta_{\mu\nu}, \qquad T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}^{\mu\nu} = \mathcal{E}u^\mu u^\nu - (\mathcal{P}+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$\begin{cases} u_\nu\partial_\mu T^{\mu\nu} = 0 \\ \partial_\mu T^{\mu\langle\nu\rangle} = 0 \end{cases} \Rightarrow \begin{cases} \dot{\mathcal{E}} = -\theta(\mathcal{E}+\mathcal{P}+\Pi) + \pi^{\mu\nu}\sigma_{\mu\nu} \\ (\mathcal{E}+\mathcal{P}+\Pi)\,\dot{u}^\nu = \nabla^\nu(\mathcal{P}+\Pi) - \nabla_\mu\pi^{\mu\langle\nu\rangle} + \pi^{\nu\alpha}\dot{u}_\alpha \end{cases}$$

$$\dot{\mathcal{P}}^{\langle\mu\rangle\langle\nu\rangle} + C_{-1}^{\langle\mu\rangle\langle\nu\rangle} = 2(\mathcal{P}+\Pi)\sigma^{\mu\nu} + \frac{5}{3}\theta(\mathcal{P}+\Pi)\Delta^{\mu\nu} - \frac{5}{3}\theta\pi^{\mu\nu} - 2\pi_\alpha^{(\mu}\sigma^{\nu)\alpha} + 2\pi_\alpha^{(\mu}\omega^{\nu)\alpha}$$
$$-\nabla_\alpha \mathfrak{f}_{-1}^{\alpha\langle\mu\rangle\langle\nu\rangle} - \left(\sigma_{\alpha\beta} + \frac{1}{3}\theta\Delta_{\alpha\beta}\right)\mathfrak{f}_{-2}^{\alpha\beta\mu\nu}$$



The method of moments converges fast

different treatment of the residual moments

$$\rho_{-2}^{\mu_1\cdots\mu_4} \to \rho_{-2}^{\mu_1\cdots\mu_4}\Big|_{eq.}$$

$$\rho_{-1}^{\mu_1\cdots\mu_4} \neq \rho_{-2}^{\mu_1\cdots\mu_4}\Big|_{eq.}$$

L T, G Vujnovich, J Noronha, U Heinz **arXiv:1808.06436**

L T, G Vujnovich **(WIP)**

Generalization?

multiple particle species

$$\Theta(p_0)\delta(p^2-m^2)f \to \sum_i \Theta(p_0)\delta(p^2-m_i^2)f_i$$

$$C[f] \to \sum_i C_i[f_1,\cdots,f_n]$$

long range interactions (**not-immediate**)

$$p\cdot\partial f \to p\cdot\partial f + F\cdot\partial_{(p)}f$$

Mild divergencies at higher orders, due to the coupling to external fields

L T, G Vujnovich, J Noronha, U Heinz **arXiv:1808.06436**

Wigner distribution (**quantum**)

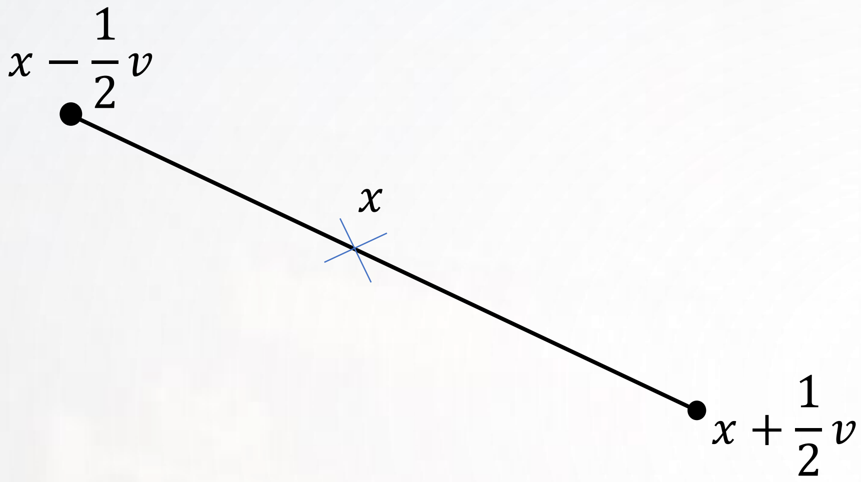
$$\Theta(p_0)\delta(p^2-m^2)f \to W$$

$$p\cdot\partial f \to k\cdot\partial W$$

Needs regularization from the start

L T, **arXiv:2003.09268**

The link between uantum fields and relativistic kinetic theory



$$W(x,k) = 4\int d^4v\, e^{-ik\cdot v}\left\langle \Phi^\dagger\,(x - 1/2v)\Phi(x + 1/2v)\right\rangle$$

$$T^{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4}\, k^\mu k^\nu W(x,k)$$

$$W(x,k) \longrightarrow (2\pi)\delta(p^2\ + m^2) f_{eq.}(x,p)$$

$$k\cdot\partial W(x,k) = \cdots \longrightarrow p\cdot\partial f(x,p) = \cdots$$

- Relativistic Kinetic Theory. Principles and Applications - De Groot, S.R. et al. Amsterdam, Netherlands: North-holland (1980)

Problem extending to the Wigner distribution

$$\delta(p^2 - m^2) f \to W$$

$$p \cdot \partial f \to k \cdot \partial W$$

different physical situations
very similar to kinetic theory but…

…ill defined from the start

$$\dot{T}^{\mu\nu} + C^{\mu\nu}_{-1} = -\nabla_\alpha \mathcal{F}^{\alpha\mu\nu}_{-1} - \nabla_\alpha u_\beta \, \mathcal{F}^{\alpha\beta\mu\nu}_{-2}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^\alpha k^\mu k^\nu}{(k \cdot u)} W = \int \frac{d^4k}{(2\pi)^4} \frac{k^{\langle\alpha\rangle} k^{\langle\mu\rangle} k^{\langle\nu\rangle}}{(k \cdot u)} W + 3u^{(\alpha} T^{\mu\nu)} - 2\mathcal{E} u^\alpha u^\mu u^\nu$$

…similar situation for the rank four tensor…

Resummed moments

Approach introduced for the Boltzmann-vlasov equation helps

$$\phi_n^{\mu_1\cdots\mu_s}(x,\zeta) = \int \frac{d^4k}{(2\pi)^4}\,(k\cdot u)^n\, e^{-\zeta(k\cdot u)^2} k^{\langle\mu_1\rangle}\dots k^{\langle\mu_s\rangle} W(x,k)$$

$$\partial_\zeta \phi_n^{\mu_1\cdots\mu_s} = -\phi_{n+2}^{\mu_1\cdots\mu_s}$$

$$\int_\zeta^\infty dv\, \phi_{n+2}^{\mu_1\cdots\mu_s} = \phi_n^{\mu_1\cdots\mu_s}$$

$$\phi_n^{\mu_1\cdots\mu_s}(x,0) = \Delta_{\boldsymbol{\alpha_1}}^{\boldsymbol{\mu_1}}\cdots\Delta_{\boldsymbol{\alpha_s}}^{\boldsymbol{\mu_s}} \mathcal{F}_{\mathrm{n}}^{\alpha_1\cdots\alpha_s} = \mathfrak{f}_n^{\alpha_1\cdots\alpha_s}$$

All (well-defined) previous moments recovered from the resumed ones, including $T^{\mu\nu}$

2 generations of dynamical moments needed

$$\dot{\phi}_2^{\langle\mu_1\rangle\cdots\langle\mu_1\rangle} + \tilde{C}_1^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = -\theta\phi_2^{\mu_1\cdots\mu_s} - s\nabla_\alpha u^{(\mu_1}\phi_2^{\mu_2\cdots\mu_s)\alpha} - \nabla_\alpha\phi_1^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + \dot{u}_\alpha\left[2\phi_1^{\alpha\mu_1\cdots\mu_s} + 2\zeta\partial_\zeta\phi_1^{\alpha\mu_1\cdots\mu_s}\right]$$
$$-s\dot{u}^{(\mu_1}\partial_\zeta\phi_1^{\mu_2\cdots\mu_s)} + \nabla_\alpha u_\beta\left[\int_\zeta^\infty dv\,\phi_2^{\alpha\mu_1\cdots\mu_s} - 2\zeta\phi_2^{\alpha\mu_1\cdots\mu_s}\right]$$

$$\dot{\phi}_1^{\langle\mu_1\rangle\cdots\langle\mu_1\rangle} + \tilde{C}_0^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = -\theta\phi_1^{\mu_1\cdots\mu_s} - s\nabla_\alpha u^{(\mu_1}\phi_1^{\mu_2\cdots\mu_s)\alpha} - \nabla_\alpha\int_\zeta^\infty dv\phi_1^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + \dot{u}_\alpha\left[\int_\zeta^\infty dv\,\phi_2^{\alpha\mu_1\cdots\mu_s} - 2\zeta\phi_2^{\alpha\mu_1\cdots\mu_s}\right]$$
$$+s\dot{u}^{(\mu_1}\phi_2^{\mu_2\cdots\mu_s)} - 2\zeta\nabla_\alpha u_\beta\phi_1^{\alpha\beta\mu_1\cdots\mu_s}$$

Exactly solvable case

Bjorken symmetry

$$\tau = \sqrt{t^2 - z^2}, \qquad v = k^0 t - z\,k^z, \qquad u = (\cosh\eta, 0, 0, \sinh\eta)$$

$$\eta = \frac{1}{2}\ln\left(\frac{t+z}{t-z}\right), \qquad w = zk^0 - t\,k^z, \qquad z = (\sinh\eta, 0, 0, \cosh\eta)$$

(as a consequence)

$$T^{\mu\nu} = \mathcal{E}(\tau)u^\mu u^\nu + \mathcal{P}_T(\tau)\,(x^\mu x^\nu + y^\mu y^\nu) + \mathcal{P}_L(\tau) z^\mu z^\nu$$

$$\pi^{\mu\nu} = -\frac{1}{2}\pi(\tau)\,(x^\mu x^\nu + y^\mu y^\nu) + \pi(\tau) z^\mu z^\nu$$

$$\mathcal{P}_T = \mathcal{P} + \Pi - \frac{1}{2}\pi, \qquad \mathcal{P}_L = \mathcal{P} + \Pi + \pi$$

RTA

$$k\cdot\partial W = -\frac{k\cdot u}{\tau_R}\left(W - W_{eq}\right) = -\frac{k\cdot u}{\tau_R}\left(W - \frac{2\delta(k^2)}{(2\pi)^3} e^{-\frac{1}{T(\tau)}\sqrt{k_T^2 + \frac{w^2}{\tau^2}}}\right) \Rightarrow \partial_\tau W + 2\frac{v^2 - w^2}{\tau}\partial v^2 W = \frac{1}{\tau_R}\delta W$$

in addition $\quad W(\tau, v^2, k_T, w^2)$

Hydrodynamic expansion

$$L_n = \phi_2^{\mu_1\cdots\mu_{2n}} z_{\mu_1}\cdots z_{\mu_{2n}}, \qquad T_n = \phi_2^{\mu_1\cdots\mu_{2n}\alpha\beta} z_{\mu_1}\cdots z_{\mu_{2n}} x_\alpha x_\beta$$

$$\dot{L}_n + \frac{1}{\tau_R}(L_n - L_n^{eq.}) = -\frac{2n+1}{\tau} L_n + \frac{1}{\tau}\hat{\mathcal{L}} L_{n+1}$$

$$\dot{T}_n + \frac{1}{\tau_R}(T_n - T_n^{eq.}) = -\frac{2n+1}{\tau} T_n + \frac{1}{\tau}\hat{\mathcal{L}} T_{n+1}$$

$$\mathcal{E} = L_0(\tau, \zeta = 0)$$

$$\mathcal{P}_L = \int_\zeta^\infty d\zeta' \, L_1(\tau, \zeta')$$

$$\mathcal{P}_T = \int_\zeta^\infty d\zeta' \, T_0(\tau, \zeta')$$

$$\hat{\mathcal{L}}\,[f] = 2\zeta\, f(\zeta) - \int_\zeta^\infty d\zeta' f(\zeta')$$

Hydrodynamics

$$\dot{\mathcal{E}} = -\frac{\mathcal{E}+\mathcal{P}_L}{\tau}$$

$$\dot{\mathcal{P}}_L + \frac{1}{\tau_R}\left(\mathcal{P}_L - \frac{1}{3}\mathcal{E}\right) = -\frac{3}{\tau}\mathcal{P}_L + \frac{1}{\tau}\mathcal{R}_L^{(1)}$$

$$\dot{\mathcal{P}}_T + \frac{1}{\tau_R}\left(\mathcal{P}_T - \frac{1}{3}\mathcal{E}\right) = -\frac{1}{\tau}\mathcal{P}_L + \frac{1}{\tau}\mathcal{R}_T^{(1)}$$

one can integrate the equations in ζ

…the same for the sources and their equations…

Hydrodynamic expansion

Hydrodynamics

$$\dot{\mathcal{E}} = -\frac{\mathcal{E}+\mathcal{P}_L}{\tau}$$

$$\dot{\mathcal{P}}_L + \frac{1}{\tau_R}\left(\mathcal{P}_L - \frac{1}{3}\mathcal{E}\right) = -\frac{3}{\tau}\mathcal{P}_L + \frac{1}{\tau}\mathcal{R}_L^{(1)}$$

$$\dot{\mathcal{P}}_T + \frac{1}{\tau_R}\left(\mathcal{P}_T - \frac{1}{3}\mathcal{E}\right) = -\frac{1}{\tau}\mathcal{P}_L + \frac{1}{\tau}\mathcal{R}_T^{(1)}$$

$$\hat{\mathcal{L}}\,[f] = 2\zeta\, f(\zeta) - \int_\zeta^\infty d\zeta' f(\zeta')$$

systematically improvable set of scalar equations…

$$\mathcal{R}_T^{(n)} = \int_0^\infty d\zeta\, \left(\hat{\mathcal{L}}\right)^n T_n\,, \qquad \mathcal{R}_L^{(n)} = \int_0^\infty d\zeta\, \left(\hat{\mathcal{L}}\right)^n L_{n+1}$$

$$\dot{\mathcal{R}}_T^{(n)} + \frac{1}{\tau_R}\delta\mathcal{R}_T^{(n)} = -\frac{2n+1}{\tau}\mathcal{R}_T^{(n)} + \frac{1}{\tau}\mathcal{R}_T^{(n+1)}$$

$$\dot{\mathcal{R}}_L^{(n)} + \frac{1}{\tau_R}\delta\mathcal{R}_T^{(n)} = -\frac{2n+3}{\tau}\mathcal{R}_L^{(n)} + \frac{1}{\tau}\mathcal{R}_L^{(n+1)}$$

…to test against the exact solutions

Exact solutions for the wigner distribution

- Constant shear-viscosity over entropy ratio: $\tau_R = 5\bar{\eta}/T$
- $\bar{\eta} = 3/(4\pi)$
- $\tau_0 = 1/4$ fm/c, $T_0 = 0.6$ GeV, two possible initial conditions:

$$W_0^{iso} = \frac{2}{(2\pi)^3\sqrt{2\pi\sigma}} \mathrm{e}^{-\frac{v^2}{2\tau_0^2\sigma}}\, e^{-\frac{1}{T_0}\sqrt{\sigma = k_T^2 + \frac{w^2}{\tau_0^2}}} \longrightarrow \mathcal{P}_0 = \mathcal{P}_{\text{eq.}} = \frac{1}{3}\,\mathcal{E}$$

$$W_0^{a} = \frac{2}{(2\pi)^3\sqrt{2\pi\sigma}} \mathrm{e}^{-\frac{v^2}{2\tau_0^2\sigma}}\, e^{-\frac{1}{T_0}\sqrt{\sigma = k_T^2 + \frac{w^2}{\tau_0^2}}} [1 - 3P_2\left(\frac{w}{\tau_0\sqrt{\sigma}}\right)] \longrightarrow \begin{array}{l} \mathcal{P}_T^0 = \frac{8}{5}\mathcal{P}_{\text{eq.}} \\ \mathcal{P}_L^0 = -\frac{1}{5}\mathcal{P}_{\text{eq.}} \end{array}$$

Hydrodynamics

$$\dot{\mathcal{E}} = -\frac{\mathcal{E}+\mathcal{P}_L}{\tau}$$

$$\dot{\mathcal{P}}_L + \frac{1}{\tau_R}\left(\mathcal{P}_L - \frac{1}{3}\mathcal{E}\right) = -\frac{3}{\tau}\mathcal{P}_L + \frac{1}{\tau}\mathcal{R}_L^{(1)}\Big|_{eq}$$

$$\dot{\mathcal{P}}_T + \frac{1}{\tau_R}\left(\mathcal{P}_T - \frac{1}{3}\mathcal{E}\right) = -\frac{1}{\tau}\mathcal{P}_T + \frac{1}{\tau}\mathcal{R}_T^{(1)}\Big|_{eq}$$

What can we say for the isotropic case

$$\mathrm{R}_L^{eq.} = \frac{1}{5}\mathcal{E} \qquad \mathrm{R}_L^{0} = -\frac{1}{5}\mathcal{E}$$

$$\mathrm{R}_T^{eq.} = \frac{1}{15}\mathcal{E} \qquad \mathrm{R}_T^{0} = -\frac{1}{15}\mathcal{E}$$

$$\frac{\delta\dot{\mathcal{P}}_L}{\dot{\mathcal{P}}_L}\Big|_0 = -\frac{1}{3}$$

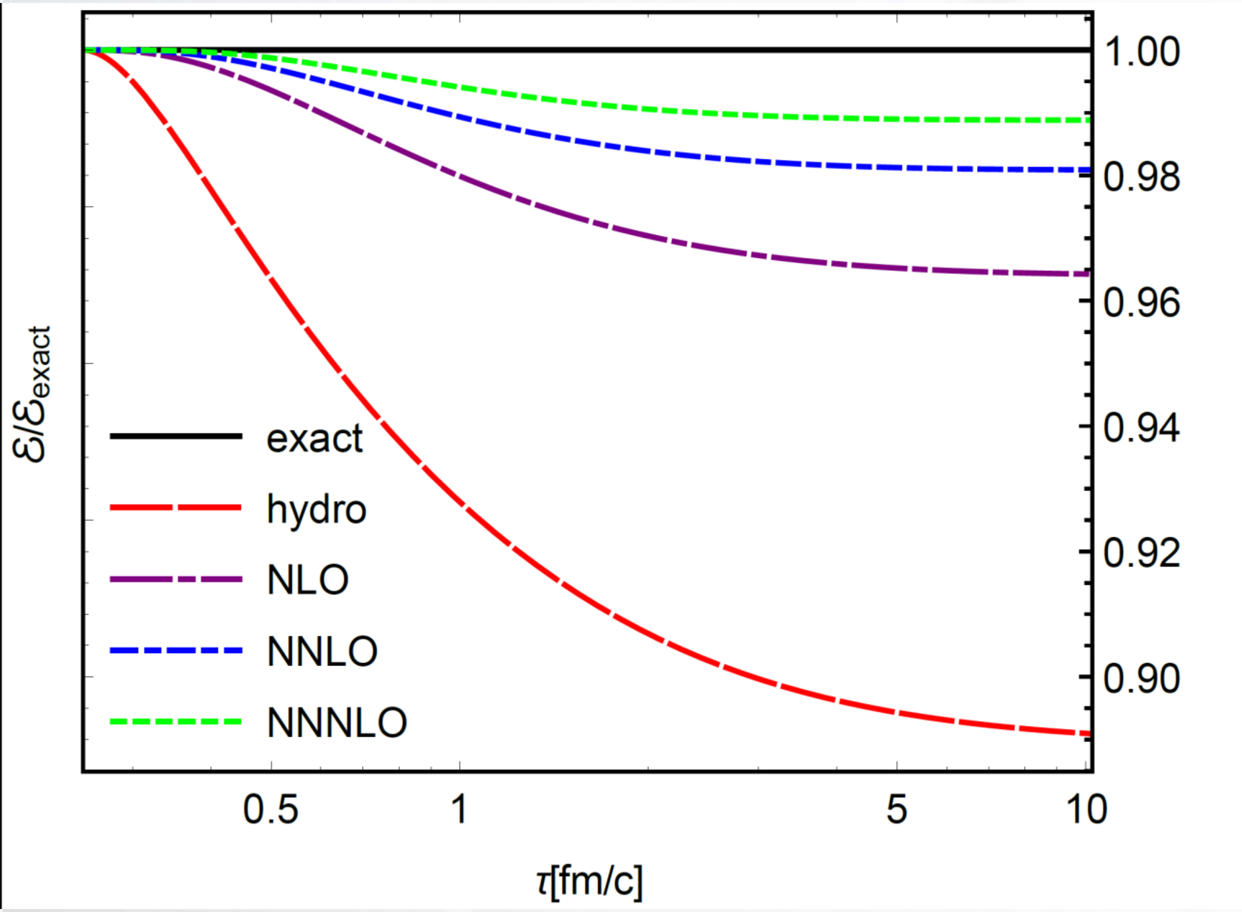
$$\frac{\delta\dot{\mathcal{P}}_T}{\dot{\mathcal{P}}_T}\Big|_0 = -\frac{1}{3}$$

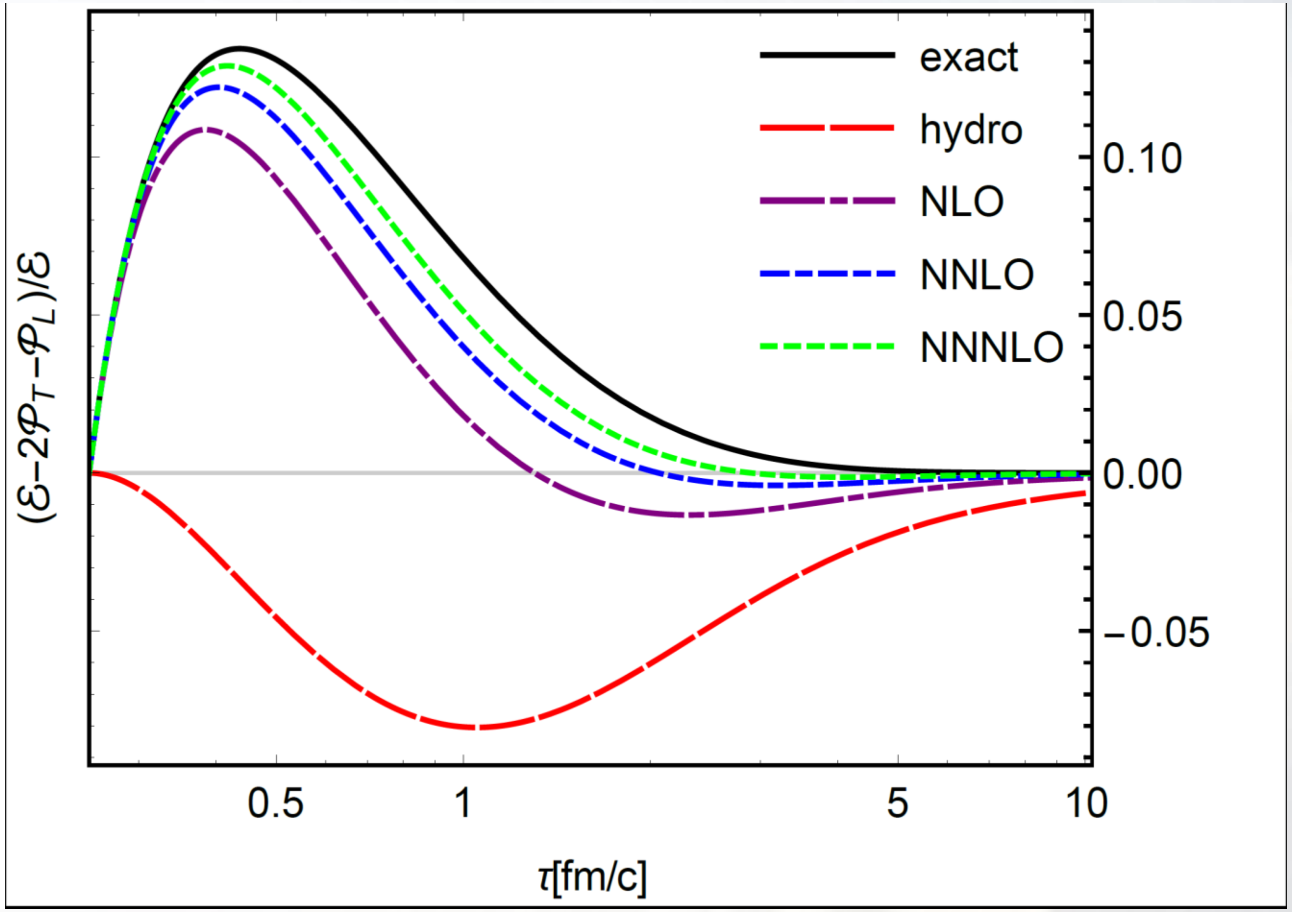
$$\delta\mathcal{P}_L = \int_{\tau_0}^{\tau} ds\, \delta\dot{\mathcal{P}}_L \Rightarrow \frac{\delta\mathcal{P}_L}{\mathcal{P}_L} = \frac{\int \delta\dot{\mathcal{P}}_L}{\mathcal{P}_L} \Rightarrow \text{Maximum if } 0 = \partial_\tau\left(\frac{\delta\mathcal{P}_L}{\mathcal{P}_L}\right) = \frac{\delta\dot{\mathcal{P}}_L}{\mathcal{P}_L} - \frac{\delta\mathcal{P}_L}{\mathcal{P}_L}\frac{\dot{\mathcal{P}}_L}{\mathcal{P}_L} \Rightarrow \frac{\delta\mathcal{P}_L}{\mathcal{P}_L} = \frac{\delta\dot{\mathcal{P}}_L}{\dot{\mathcal{P}}_L}$$

$$\frac{\delta\mathcal{E}}{\mathcal{E}} = \frac{\delta\dot{\mathcal{E}}}{\dot{\mathcal{E}}} = \frac{\delta\mathcal{E}+\delta\mathcal{P}_L}{\mathcal{E}+\mathcal{P}_L} \Rightarrow \frac{\delta\mathcal{E}}{\mathcal{E}} \simeq \frac{\delta\mathcal{P}_L}{\mathcal{P}_L}$$

...but forthe trace anomaly $\mathcal{E} - 2\mathcal{P}_T - \mathcal{P}_L = -3\Pi$ $\qquad \frac{\delta\dot{\Pi}}{\dot{\Pi}} = -1$

Comparisons with the exact solutions

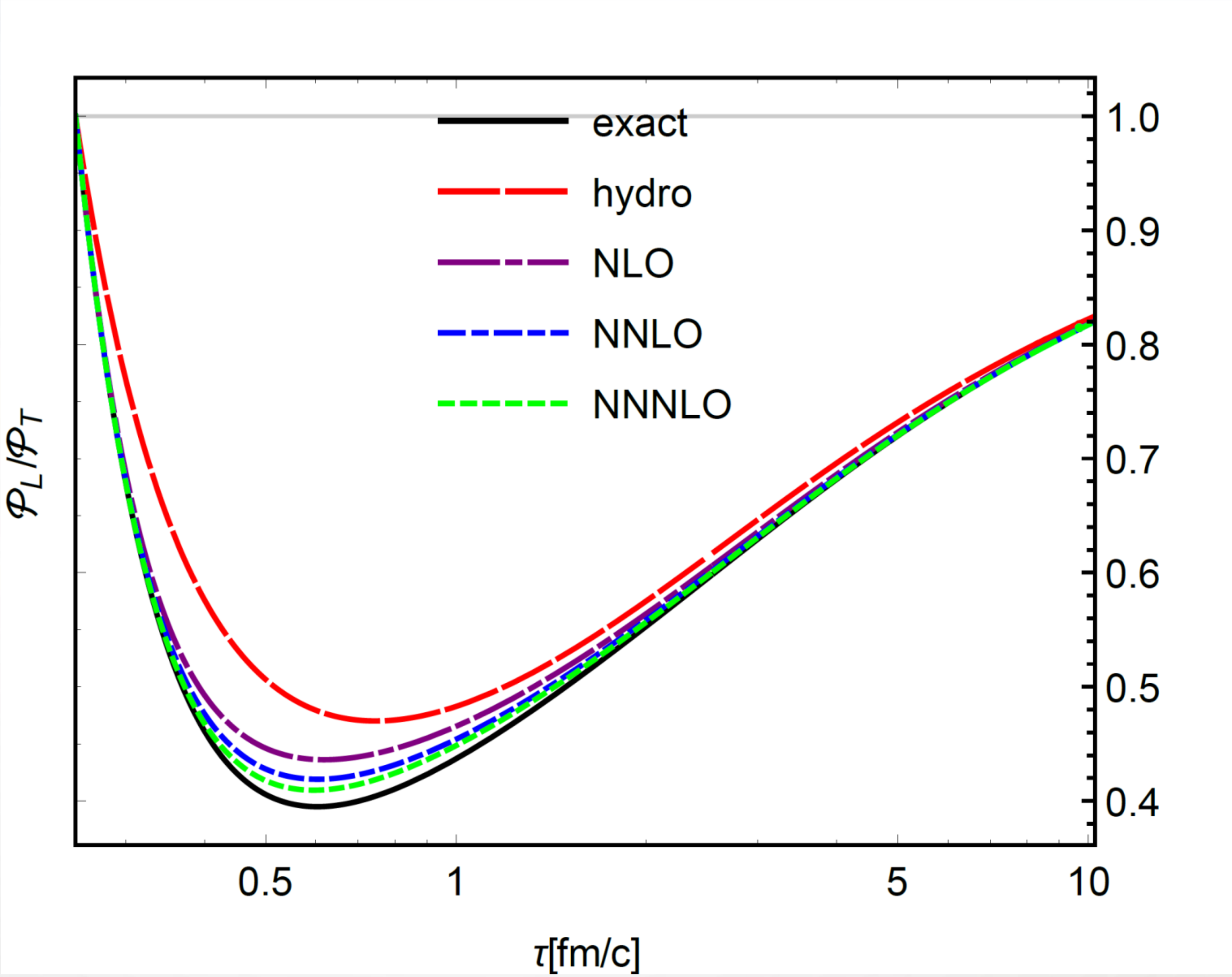




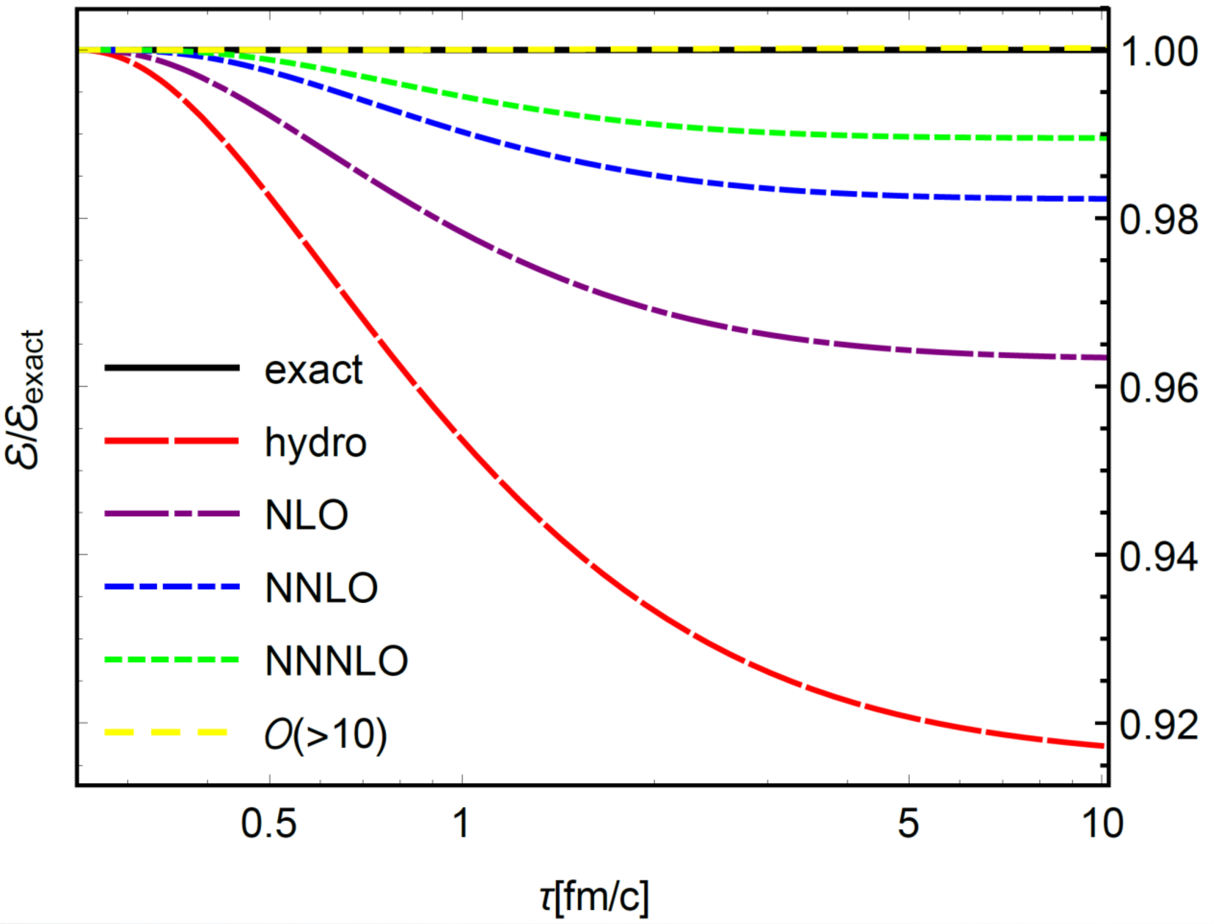
$$(\mathcal{E}-2\mathcal{P}_T-\mathcal{P}_L)/\mathcal{E}=-\frac{3\Pi}{\mathcal{E}}=-\frac{\Pi}{\mathcal{P}}$$

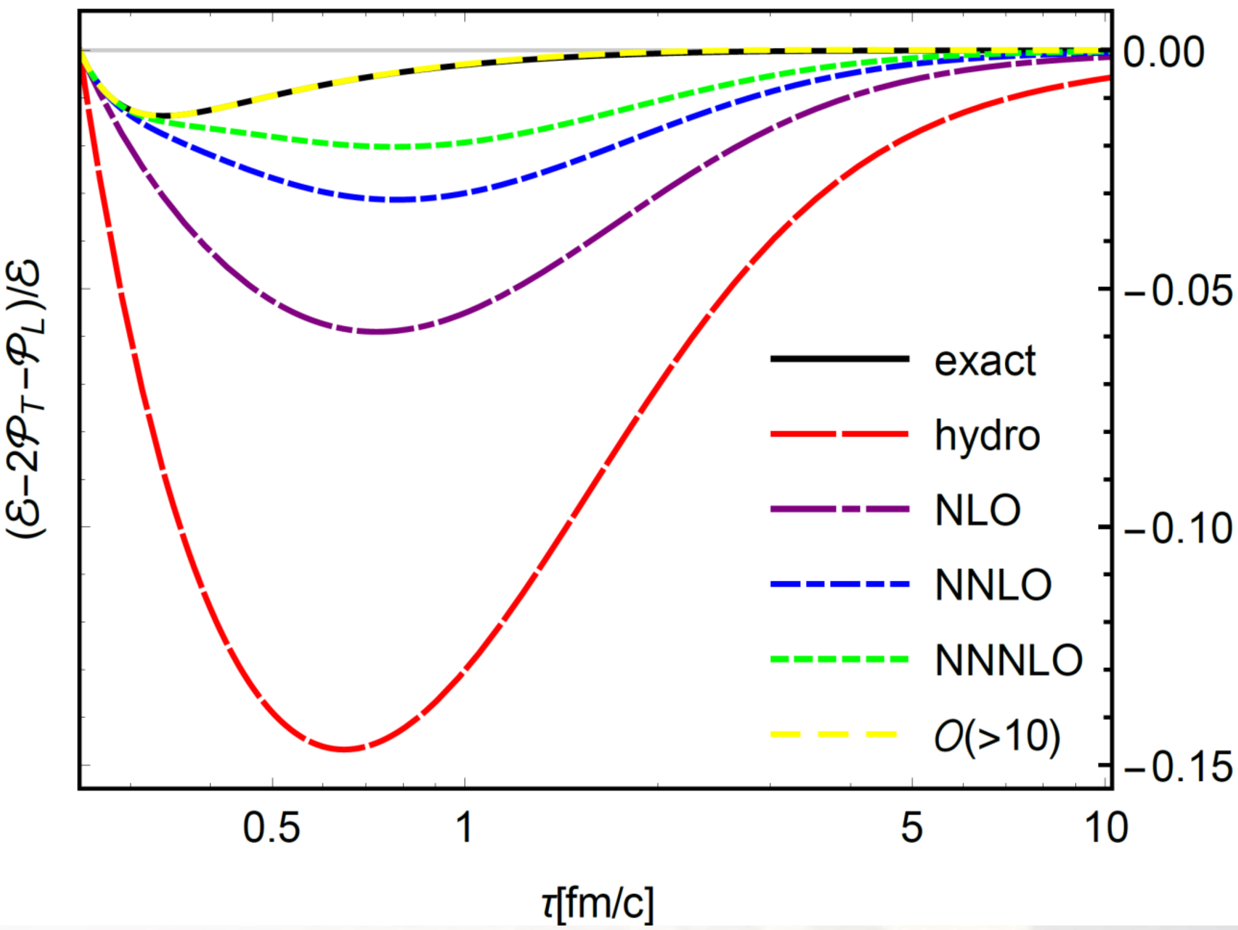
Comparisons with the exact solutions

fast convergence for the pressure anisotropy too



Comparisons for the anisotropic initial conditions

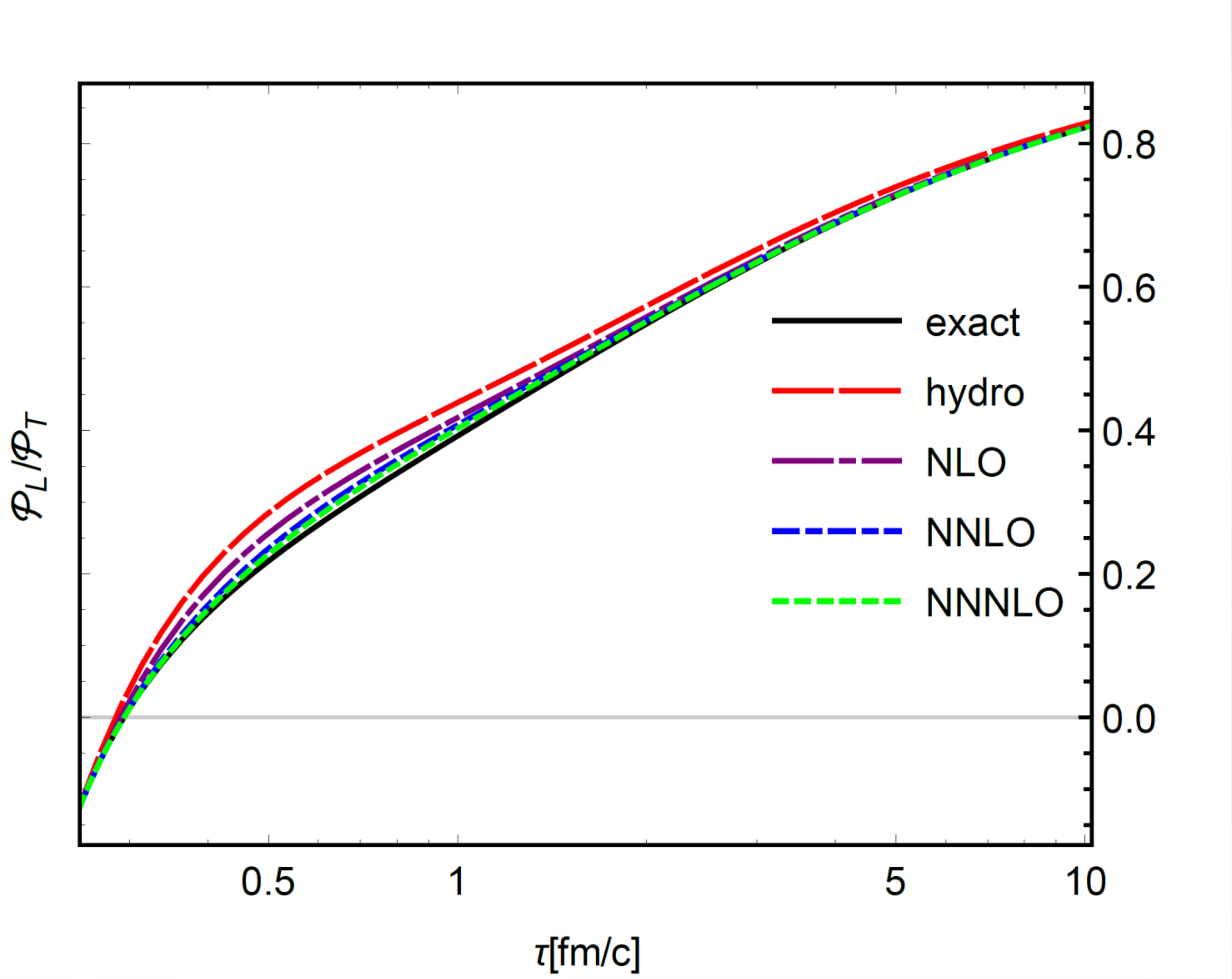




similar conclusions

Comparisons for the anisotropic initial conditions

reasonable approximation for the pressure anisotropy from the start



similar conclusions

Conclusions and outlook

- The metod of moments can't be immediately generalized to the Wigner distribution

- An expansion around the resummed moments is well defined

- Estimates of the errors for hydrodynamics, *Fast convergence?*

Back up slides

$$\int [g(x)+h(x)]\,dx \neq \int g(x)dx + \int h(x)dx$$

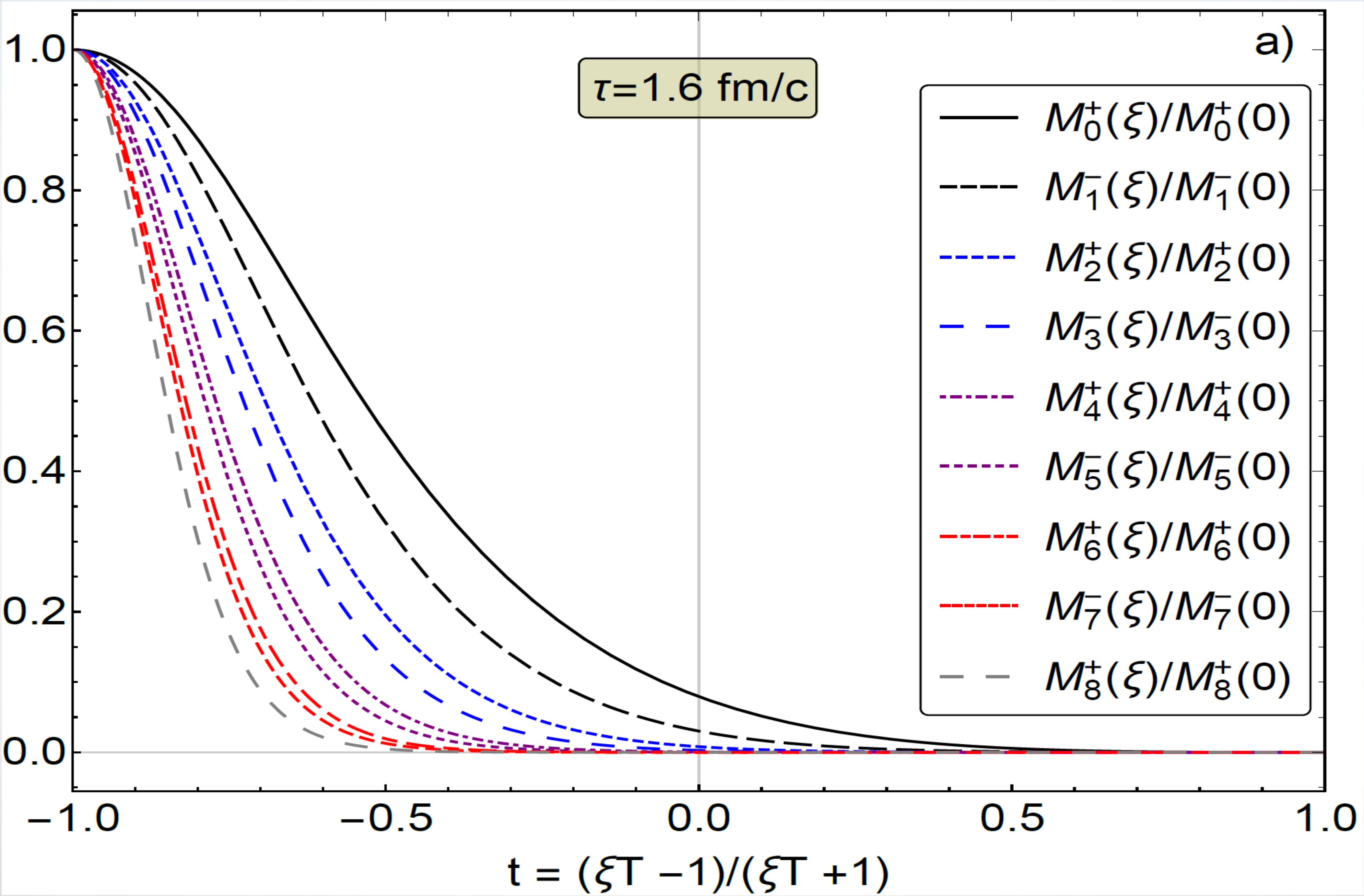
$$\int \lim_{\varepsilon\to 0} f(\varepsilon,x)dx \neq \lim_{\varepsilon\to 0}\int f(\varepsilon,x)dx$$

$$\frac{1}{\beta} = \int_0^\infty \left[-\partial_\beta\left(\frac{e^{-\beta x}}{x}\right)\right]dx \neq -\partial_\beta\left(\int_0^\infty \frac{e^{-\beta x}}{x}dx \equiv \infty\right)$$

$$\frac{1}{x} = \int_0^\infty e^{-\alpha x}d\alpha$$

$$\frac{1}{(\alpha+\beta)^2} = \int_0^\infty dx\left[-\partial_\beta\left(e^{-(\alpha+\beta)x}\right)\right] = -\partial_\beta\left(\int_0^\infty dx\, e^{-(\alpha+\beta)x} = \frac{1}{\alpha+\beta}\right),$$

$$\int_0^\infty d\alpha\left[\frac{1}{(\alpha+\beta)^2} = \partial_\alpha\left(-\frac{1}{\alpha+\beta}\right)\right] = \frac{1}{\beta}$$

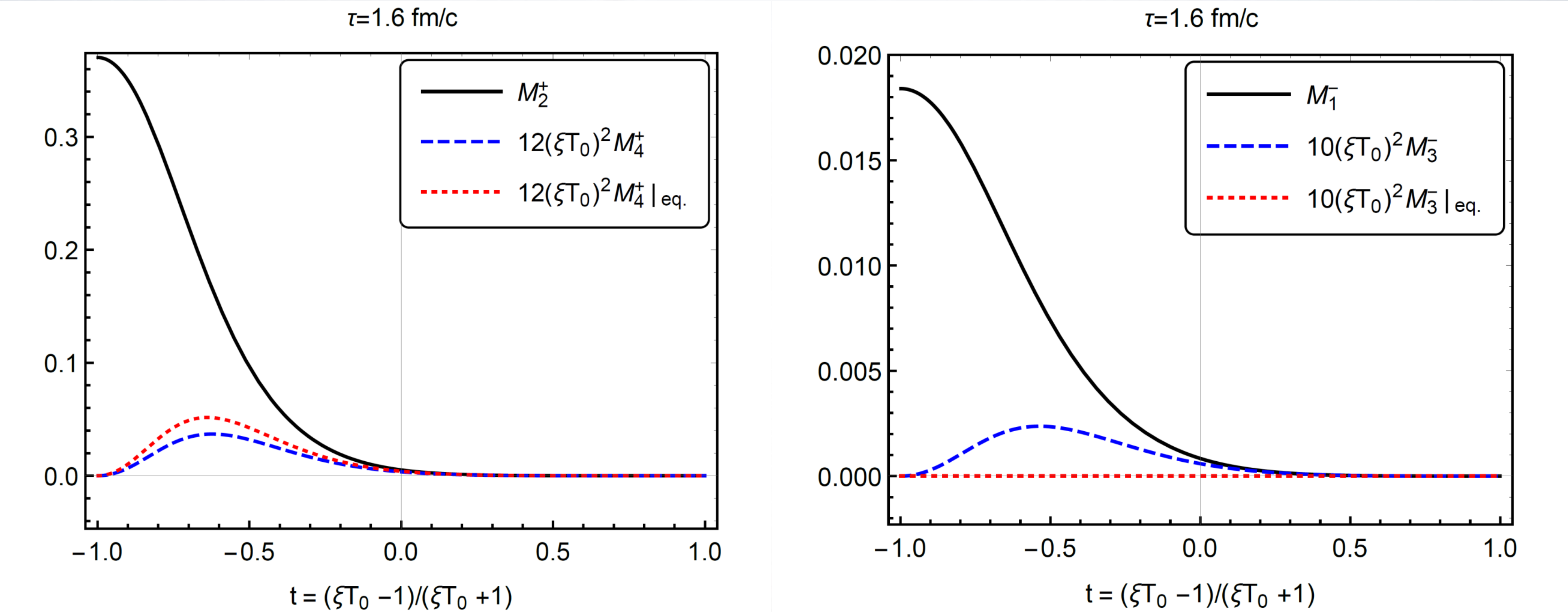


a)
τ=1.6 fm/c
$M_0^+(\xi)/M_0^+(0)$
$M_1^-(\xi)/M_1^-(0)$
$M_2^+(\xi)/M_2^+(0)$
$M_3^-(\xi)/M_3^-(0)$
$M_4^+(\xi)/M_4^+(0)$
$M_5^-(\xi)/M_5^-(0)$
$M_6^+(\xi)/M_6^+(0)$
$M_7^-(\xi)/M_7^-(0)$
$M_8^+(\xi)/M_8^+(0)$
1.0
0.8
0.6
0.4
0.2
0.0
−1.0
−0.5
0.0
0.5
1.0
t = (ξT −1)/(ξT +1)

Suppression for small ξ

$$\xi T_0 = \frac{1+t}{1-t} \Rightarrow t = \frac{\xi\, T_0 - 1}{\xi T_0 + 1}$$

$\tau = 1.6$ fm
$E_L^0/T_0 = 0.2\ \mathrm{fm}^{-1}$



from the EOM: $\tau_R \partial_\tau M_l^\pm = -\frac{\tau_R}{\tau}\left[(l+1)M_l^\pm - 2(\xi T_0)^2 (l+4)(l+1) M_{l+2}^\pm + \cdots\right.$

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu}$$

$$f_r^{\mu_1\cdots\mu_s} = \mathcal{F}_r^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}$$

$$\phi_r^{\mu_1\cdots\mu_s} = \Phi_r^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}$$

$$\begin{aligned}
\dot{f}_r^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + (\mathcal{F}_{\mathrm{coll.}})_r^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = & -q\,s\,\varepsilon^{\rho\sigma\alpha(\mu_1} f_{r-1}^{\mu_2\cdots\mu_s)\beta} g_{\alpha\beta}u_\rho B_\sigma - q(r-1)\,E_\alpha\, f_{r-2}^{\alpha\mu_1\cdots\mu_s} - q\,s\,E^{(\mu_1} f_r^{\mu_2\cdots\mu_s)} \\
& + m\dot{m}\,(r-1)\,f_{r-2}^{\mu_1\cdots\mu_s} + s\,m\nabla^{(\mu_1} m\, f_{r-1}^{\mu_2\cdots\mu_s)} \\
& + r\,\dot{u}_\alpha f_{r-1}^{\alpha\mu_1\cdots\mu_s} - s\dot{u}^{(\mu_1} f_{r+1}^{\mu_2\cdots\mu_s)} \\
& - \nabla_\alpha f_{r-1}^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle} - \theta\, f_r^{\mu_1\cdots\mu_s} - s\,\nabla_\alpha u^{(\mu_1} f_r^{\mu_2\cdots\mu_s)\alpha} \\
& + (r-1)\nabla_\alpha u_\beta\, f_{r-2}^{\alpha\beta\mu_1\cdots\mu_s},
\end{aligned}$$

$$\begin{aligned}
\dot{\phi}_1^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + (\Phi_{\mathrm{coll.}})_1^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = & -q\left[s\,E^{(\mu_1}\phi_1^{\mu_2\cdots\mu_s)} - 2\xi^2\left(E_\alpha\,\phi_1^{\alpha\mu_1\cdots\mu_s} + m\dot{m}\,\phi_1^{\mu_1\cdots\mu_s}\right)\right] \\
& + s\frac{1}{\sqrt{\pi}}\int_{\xi^2}^{\infty}\frac{d\upsilon}{\sqrt{\upsilon-\xi^2}}\left[m\nabla^{(\mu_1} m\,\phi_1^{\mu_2\cdots\mu_s)} - q\,\varepsilon^{\rho\sigma\alpha(\mu_1}\phi_1^{\mu_2\cdots\mu_s)\beta} g_{\alpha\beta}u_\rho B_\sigma\right] \\
& + \frac{1}{\sqrt{\pi}}\int_{\xi^2}^{\infty}\frac{d\upsilon}{\sqrt{\upsilon-\xi^2}}\left[\dot{u}_\alpha\phi_1^{\alpha\mu_1\cdots\mu_s} + s\,\dot{u}^{(\mu_1}\partial_\upsilon\phi_1^{\mu_2\cdots\mu_s)} + 2\xi^2\,\dot{u}_\alpha\,\partial_\upsilon\phi_1^{\alpha\mu_1\cdots\mu_s} - \nabla_\alpha\phi_1^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle}\right] \\
& - \theta\,\phi_1^{\mu_1\cdots\mu_s} - s\,\nabla_\alpha u^{(\mu_1}\phi_1^{\mu_2\cdots\mu_s)\alpha} - 2\xi^2\nabla_\alpha u_\beta\,\phi_1^{\alpha\beta\mu_1\cdots\mu_s}.
\end{aligned}$$

Particles interacting with external fields

Boltzmann-Vlasov equation $$p \cdot \partial f + m\partial_\alpha m\, \partial_{(p)}^\alpha f + qF_{\alpha\beta}p^\beta \partial_{(p)}^\alpha f = -\mathcal{C}[f]$$

Immediate (but problematic) generalization

$$\begin{aligned}
\dot{\mathcal{F}}_r^{\mu_1\cdots\mu_s} + C_{r-1}^{\mu_1\cdots\mu_s} \;=\;& r\,\dot{u}_\alpha\,\mathcal{F}_{r-1}^{\alpha\mu_1\cdots\mu_s} - \nabla_\alpha \mathcal{F}_{r-1}^{\alpha\mu_1\cdots\mu_s} + (r-1)\,\nabla_\alpha u_\beta\,\mathcal{F}_{r-2}^{\alpha\beta\mu_1\cdots\mu_s} \\
& + m\dot{m}\,(r-1)\,\mathcal{F}_{r-2}^{\mu_1\cdots\mu_s} + s\, m\partial^{(\mu_1} m\,\mathcal{F}_{r-1}^{\mu_2\cdots\mu_s)} \\
& - q(r-1)\,E_\alpha\,\mathcal{F}_{r-2}^{\alpha\mu_1\cdots\mu_s} - q\, s\, g_{\alpha\beta} F^{\alpha(\mu_1}\mathcal{F}_{r-1}^{\mu_2\cdots\mu_s)\beta}
\end{aligned}$$

$$F_{\mu\nu} = E_\mu u_\nu - E_\nu u_\mu + \varepsilon_{\mu\nu\rho\sigma} u^\rho B^\sigma$$

Moments with large negative r needed, infrared catastrophe!

Unphysical moments as a source

$$\mathcal{F}_r^{\mu_1 \cdots \mu_s} = \int_{\mathbf{p}} (p \cdot u)^r \, p^{\mu_1} \cdots p^{\mu_s} f$$

$r < -2 - s$, diverging integral in the massless case

Numerical problems for small non-vanishing masses

$$\frac{\mathcal{F}_r^{\mu_1 \cdots \mu_s}}{T^{r+s+2}} \propto \left(\frac{m}{T} \right)^{r+s+2}$$

Any non-trivial coupling to an electromagnetic field introduces numerical problems at higher orders

Moments with large negative r needed, infrared catastrophe!

Resummed expansion

- Resummed moments
- All reducible moments recovered
- Well defined equations

$$\Phi^{\mu_1\cdots\mu_s} = \int_{\mathbf{p}} (p\cdot u)p^{\mu_1}\cdots p^{\mu_s} e^{-\xi^2(p\cdot u)^2} f$$

$$\mathcal{F}_n^{\mu_1\cdots\mu_l} = \frac{2}{\sqrt{\pi}}\int_0^\infty d\xi\, \Phi^{\mu_1\cdots\mu_l\nu_1\cdots\nu_n}\, u_{\nu_1}\cdots u_{\nu_n}$$

$$\begin{aligned}\dot{\Phi}^{\mu_1\cdots\mu_s} + \delta\Phi^{\mu_1\cdots\mu_s}_{\text{coll.}} = {} & \frac{2}{\sqrt{\pi}}\int_\xi^\infty d\zeta \frac{\zeta}{\sqrt{\zeta^2-\xi^2}}\Big\{\dot{u}_\alpha\,\Phi^{\alpha\mu_1\cdots\mu_s} - \nabla_\alpha\Phi^{\alpha\mu_1\cdots\mu_s} \\ & + s\left[m\partial^{(\mu_1}m\,\Phi^{\mu_2\cdots\mu_s)} - q\,g_{\alpha\beta}F^{\alpha(\mu_1}\Phi^{\mu_2\cdots\mu_s)\beta}\right]\Big\} \\ & -2\xi^2\left[\partial_\alpha u_\beta\,\Phi^{\alpha\beta\mu_1\cdots\mu_s} + m\dot{m}\,\Phi^{\mu_1\cdots\mu_s} - qE_\alpha\,\Phi^{\alpha\mu_1\cdots\mu_s}\right]\end{aligned}$$

Contribution from the collisional kernel

$$\delta\Phi^{\mu_1\cdots\mu_s}_{\text{coll.}} = \int_{\mathbf{p}} p^{\mu_1}\cdots p^{\mu_s} e^{-\xi^2(p\cdot u)^2}\mathcal{C}[f]$$

Exact solutions of the Boltzmann-Vlasov equation

- Maxwell equations, particles as the source: $\partial_\mu F^{\mu\nu} = J^\nu$ $\quad \epsilon^{\mu\nu\rho\sigma}\,\partial_\nu F_{\rho\sigma} = 0$

- Longitudinally boost invariant expansion, and homogeneous in the transverse plane (no parity invariance), RTA

Because of symmetry

$$\partial\tau f(\tau, p_T, p_\eta) = -q\, E_\eta \frac{\partial f}{\partial p_\eta} - \frac{1}{\tau_R}(f - f_{eq.}) \qquad \partial_\tau E_\eta(\tau) = \frac{1}{\tau} E_\eta - J_\eta$$

$$\partial\tau \bar{f}(\tau, p_T, p_\eta) = +q\, E_\eta \frac{\partial \bar{f}}{\partial p_\eta} - \frac{1}{\tau_R}(\bar{f} - f_{eq.}) \qquad u \cdot J = 0$$

$$f_{eq.} = k\, e^{-\frac{1}{T}(p\cdot u)} \qquad E_\eta = -\tau\, E_L \qquad k = \frac{N_{dof}}{(2\pi)^3} \qquad J_\eta = -\tau\, J_L$$

- Massless particles, $4\pi\,\bar{\eta} = 1$

- Local equilibrium initial conditions, $\tau_0 = 1$ fm/c, $\mathrm{T}_0 = 0.3$ GeV, $\mathrm{E}_L^0/T_0 = 0.2\ \mathrm{fm}^{-1}$.

R Ryblewski, W Florkowski, arXiv:1307.0356

Set of independent moments

❖ Linearly independent moments: $\Phi_l^{\pm} = \Phi^{\mu_1\ldots\mu_l} z_{\mu_1} \cdots z_{\mu_l} \pm \bar{\Phi}^{\mu_1\ldots\mu_l} z_{\mu_1} \cdots z_{\mu_l}$

$$[z^\mu = (\sinh\eta, 0,0, \cosh\eta), \quad u^\mu = (\cosh\eta, 0,0, \sinh\eta)]$$

❖ Normalized (dimensionless) moments: $$M_l^{\pm} = \frac{\Phi_l^{\pm}}{(8\pi k)(l+2)l!T_0^{l+3}}$$

In particular

$$(48\,\pi\,k)T^4 = \mathcal{E} = \frac{2}{\sqrt{\pi}}\int_0^\infty d\,\xi\,\left(-\frac{\partial}{\partial\xi^2}\Phi_0^+\right) = 32\sqrt{\pi}\,k\,T_0^3\int_0^\infty d\,\xi\,\left(-\frac{\partial}{\partial\xi^2}M_0^+\right)$$

$$\mathcal{P}_L = \frac{2}{\sqrt{\pi}}\int_0^\infty d\,\xi\,\Phi_2^+ = 128\sqrt{\pi}\,k\,T_0^5\int_0^\infty d\,\xi\,M_2^+$$

$$J_L = -q\frac{2}{\sqrt{\pi}}\int_0^\infty d\,\xi\,\Phi_1^- = -\,q(48\sqrt{\pi}\,k\,T_0^4)\int_0^\infty d\,\xi\,M_1^-$$

Equations to test numerically

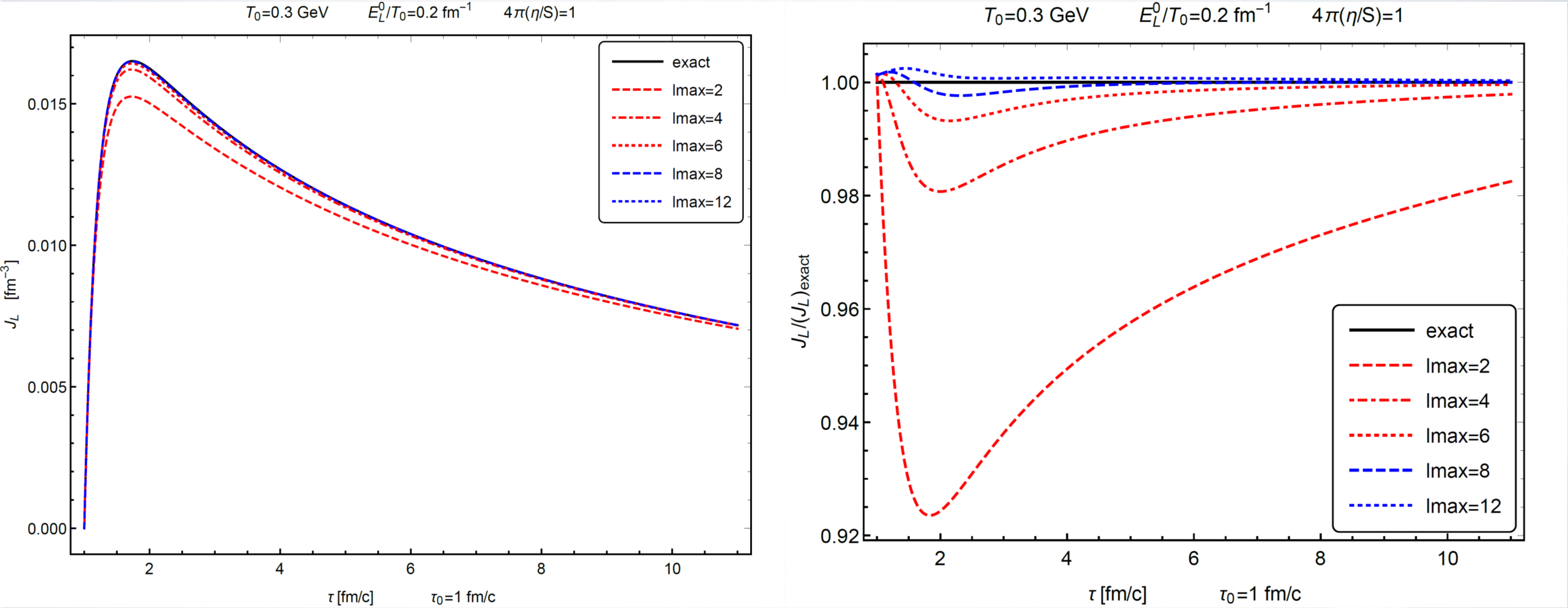
From the resummed moments and Maxwell equations

$$\tau_R \partial_\tau M_l^\pm + \left(M_l^\pm - M_l^\pm \Big|_{eq.} \right) = -\frac{\tau_R}{\tau} \Bigg[(l+1) M_l^\pm - 2(\xi T_0)^2 (l+4)(l+1) M_{l+2}^\pm + \frac{qE_\eta}{T_0} \left(\frac{l+1}{l+2} M_{l-1}^\mp - 2(\xi T_0)^2 \frac{(l+3)(l+1)}{l+2} M_{l+1}^\mp \right) \Bigg]$$

$$\frac{\partial_\tau T}{T} = -\frac{1}{4\tau} \left[1 + \frac{4T_0^5}{3\sqrt{\pi} T^4} \int_0^\infty d\xi \; M_2^+ - q\, E_\eta \frac{T_0^4}{\sqrt{\pi} T^4} \int_0^\infty d\xi \; M_1^- \right]$$

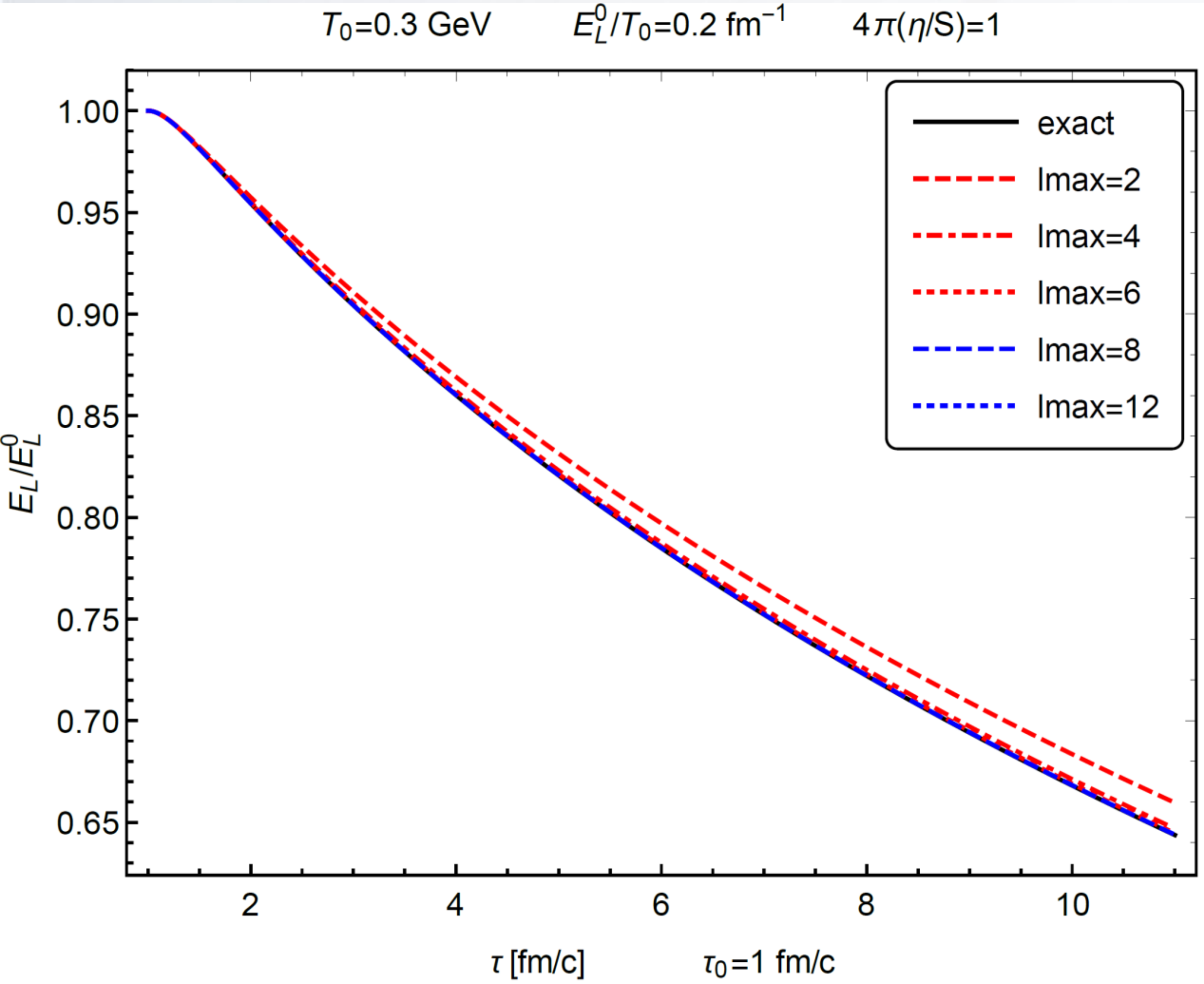
$$\partial_\tau E_\eta = \frac{1}{\tau} E_\eta \; - \tau q (48\sqrt{\pi}\, k\, T_0^4) \int_0^\infty d\,\xi \; M_1^-$$

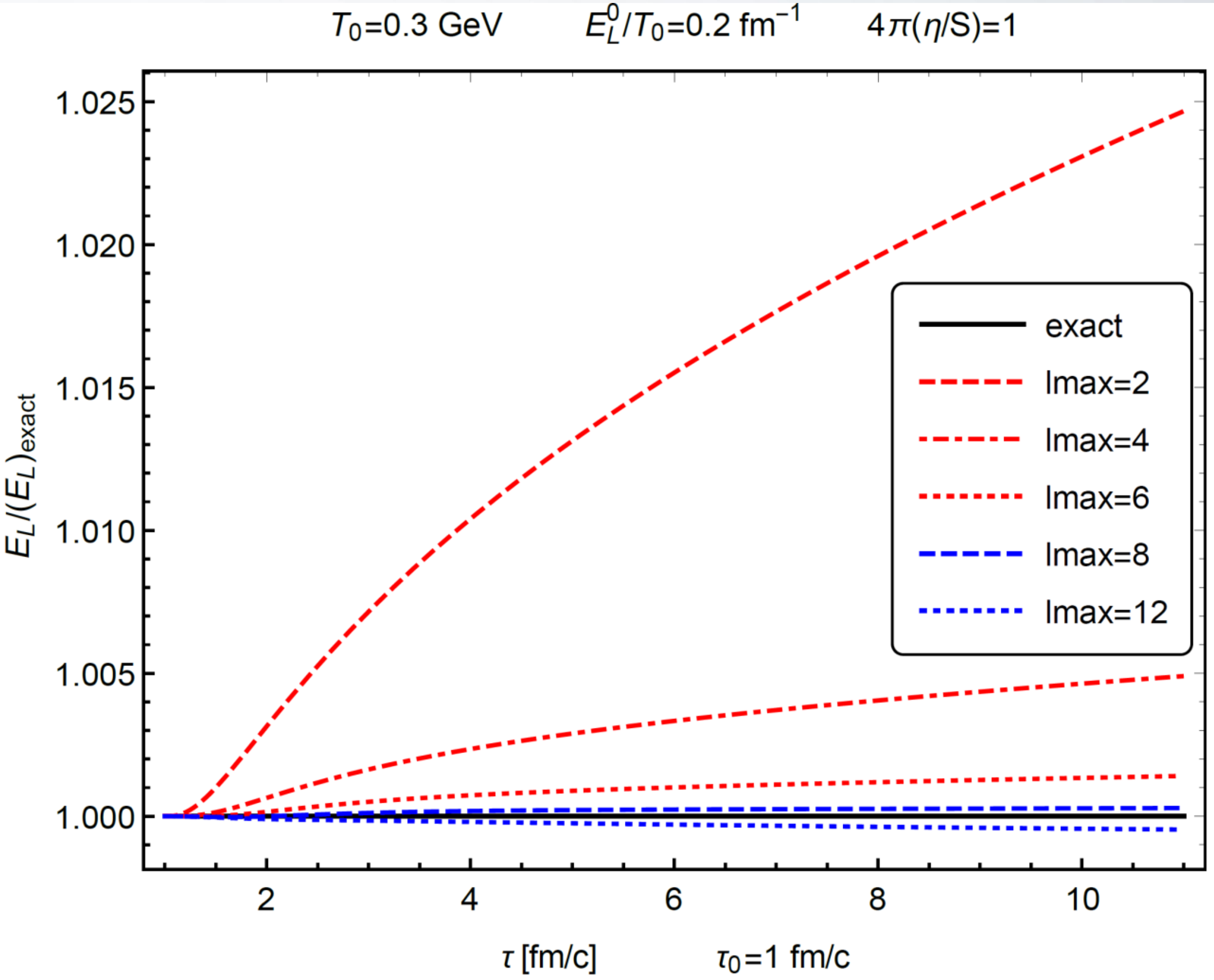
Comparisons:electric current



Higher orders ill-defined in the traditional expasion

Comparisons: electric field





Higher orders ill-defined in the traditional expasion