Search for the QCD Phase Transition with Factorial Cumulants

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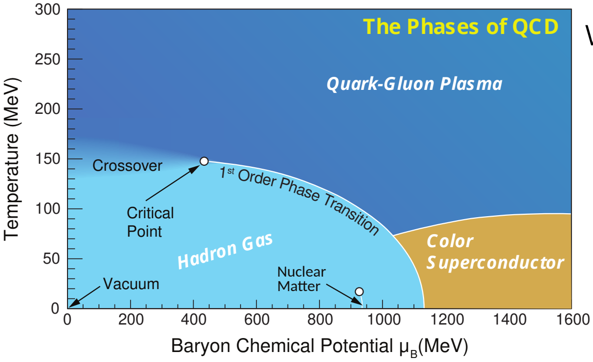
Outline

1. Introduction
   - QCD phase diagram
   - Cumulants and factorial cumulants
   - Interesting recent results
2. Our results: factorial cumulants and cumulants from baryon conservation (& short-range correlations)
   - Mathematical models
   - Analytical results
3. Summary

Introduction

The conjectured QCD phase diagram



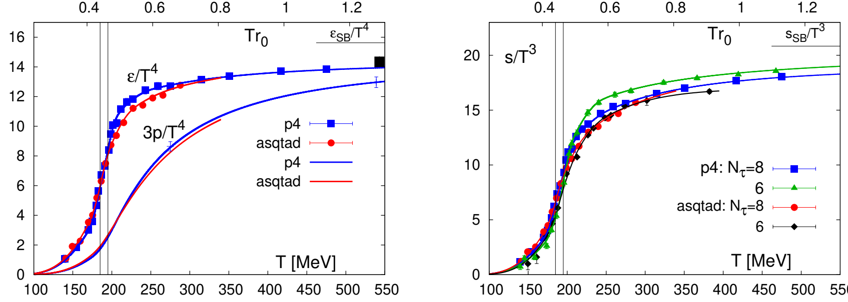
A. Bzdak, S. Esumi, V. Koch, J. Liao, M. Stephanov and N. Xu, Phys. Rept. 853, 1-87 (2020)
A. Aprahamian, A. Robert, H. Caines, *et al.*, *Reaching for the horizon: The 2015 long range plan for nuclear science*

What we know:

- nuclear matter,
- hadron gas,
- QGP - signatures from LHC and RHIC:
  - azimuthal asymmetry of particle production (elliptic flow etc.),
  - jet quenching,
  - well described by hydrodynamics,
  - strangeness enhancement.
- crossover (see next slide)

Smooth crossover at $\mu_B \approx 0$

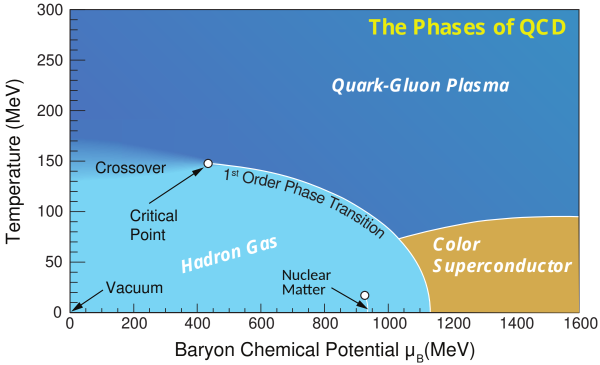
Lattice QCD results.



Y. Aoki, G. Endrodi, Z. Fodor, S. D. Katz and K. K. Szabo, Nature 443, 675-678 (2006)
A. Bazavov, T. Bhattacharya, M. Cheng, *et al.*, Phys. Rev. D 80, 014504 (2009)

The conjectured QCD phase diagram

- Most of the regions are accessible neither by QCD (non-perturbative) nor by LQCD (sign problem).
- Search for the phase transition and CEP predicted by effective models.



A. Bzdak, S. Esumi, V. Koch, J. Liao, M. Stephanov and N. Xu, Phys. Rept. 853, 1-87 (2020)
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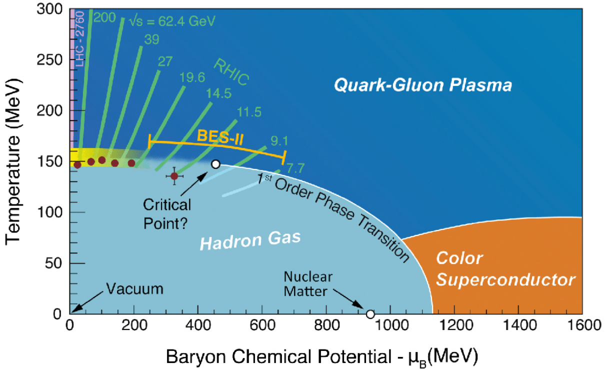
The conjectured QCD phase diagram

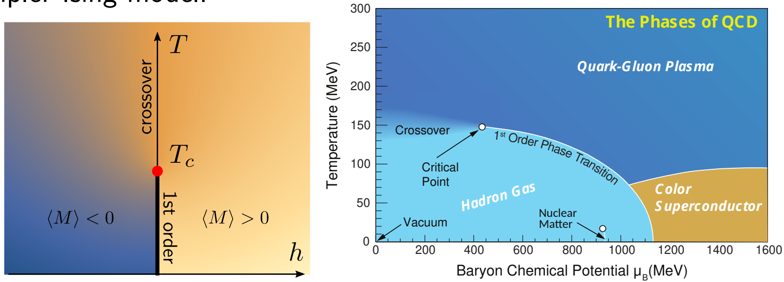
- Experiments: heavy-ion collisions at different energies.
- lower energy ⇒ higher μ_B.
- BES at RHIC, NA61/SHINE at CERN SPS.



A. Bzdak, S. Esumi, V. Koch, J. Liao, M. Stephanov and N. Xu, Phys. Rept. 853, 1-87 (2020)
A. Aprahamian, A. Robert, H. Caines, *et al.*, *Reaching for the horizon: The 2015 long range plan for nuclear science*

Critical point

Example: Ising model.



A. Bzdak, S. Esumi, V. Koch, J. Liao, M. Stephanov and N. Xu, Phys. Rept. 853, 1-87 (2020)

When approaching the critical point

- it should be an increase in fluctuations of
  - net-baryon number,
  - electric charge,
  - strangeness;
- correlation length should grow.

Phase transition

Example: water-vapor transition.

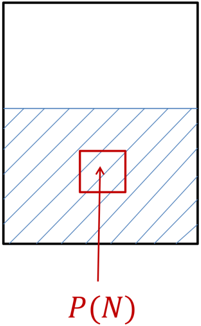
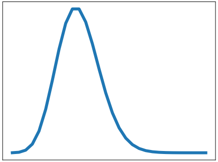
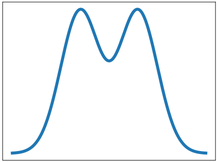


Fig: A. Bzdak

Probability distribution P(N),
N - number of H_2O molecules.



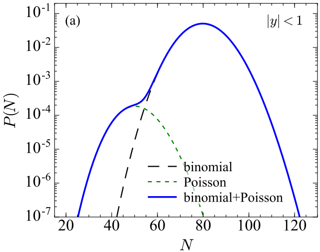
only water



phase transition: both water and vapor

Phase transition - bimodal distribution

Two-component distribution: $P(N) = (1-\alpha)P_A(N) + \alpha P_B(N)$, $0 \leq \alpha \leq 1$.



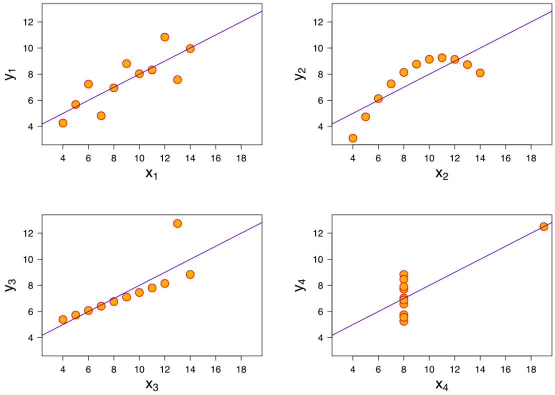
A. Bzdak, V. Koch, D. Oliinychenko and J. Steinheimer, Phys. Rev. C 98, no.5, 054901 (2018)

STAR: there is no 2-component structure at $\sqrt{s_{NN}} \geq 11.5$ GeV.
[STAR], [arXiv:2207.09837].

Characterize probability distribution

Most likely, the modifications of the probability distributions are very tiny.
We characterize the shape of $P(N)$.
The mean and the standard deviation are not enough!

- Anscombe's quartet
  F. Anscombe, (1973) Graphs in statistical analysis. American Statistician, 27, 17–21



https://en.wikipedia.org/wiki/Anscombe%27s_quartet

Cumulants and factorial cumulants

We characterize the shape of $P(N)$ by

- cumulants κ_m,
- factorial cumulants $\hat{C}_m$,
- factorial moments F_m

and use them rather than $P(N)$ itself.

Reference: Appendix of:

A. Bzdak, S. Esumi, V. Koch, J. Liao, M. Stephanov and N. Xu, Phys. Rept. 853, 1-87 (2020)

Cumulants

- Cumulant generating function: $K(t) = \ln\left[\sum_{n=0}^{\infty} P(n) e^{tn}\right]$
  Cumulants: $\kappa_m = \frac{d^m K}{dt^m}\Big|_{t=0}$

Cumulants vs central moments

- mean: $\langle N \rangle = \kappa_1$
- variance: $\sigma^2 = \langle (N - \langle N \rangle)^2 \rangle = \kappa_2$
- skewness: $S = \langle (N - \langle N \rangle)^3 \rangle / \sigma^3 = \frac{\kappa_3}{\kappa_2^{3/2}}$
- kurtosis: $K = \langle (N - \langle N \rangle)^4 \rangle / \sigma^4 - 3 = \frac{\kappa_4}{\kappa_2^2}$

Cumulant ratios

- $\frac{\kappa_2}{\kappa_1} = \frac{\sigma^2}{\langle N \rangle}$
- $\frac{\kappa_3}{\kappa_2} = S\sigma$
- $\frac{\kappa_4}{\kappa_2} = K\sigma^2$

Cumulants in statistical mechanics (in the grand-canonical ensemble)

$$\frac{\partial^n \left(P/T^4\right)}{\partial \left(\mu_B/T\right)^n} = \chi_n = \frac{\kappa_n}{VT^3} \quad \Rightarrow \quad \frac{\chi_m}{\chi_n} = \frac{\kappa_m}{\kappa_n},$$

χ_n - nth scaled susceptibility.

Factorial cumulants

- Factorial cumulant generating function: $G(z) = \ln\left[\sum_{n=0}^{\infty} P(n) z^n\right]$
  Factorial cumulants: $\hat{C}_m = \frac{d^m G}{dz^m}\Big|_{z=1}$
- Two-particle correlation function from rapidity multiplicity distributions:

$$\rho(y_1, y_2) = \rho(y_1)\rho(y_2) + C_2(y_1, y_2)$$

- Factorial cumulants are integrated correlation functions.

$$\hat{C}_2 = \int dy_1 dy_2 \; C_2(y_1, y_2)$$

- By analogy for multiparticle correlations.

Cumulants vs factorial cumulants

Cumulants

+ naturally appear in statistical mechanics and in Lattice QCD calculations,
+ often measured in experiments,
– they mix correlation functions of different orders.

Generating func: $K(t) = \ln\left[\sum_{n=0}^{\infty} P(n) e^{tn}\right]$

Cumulants: $\kappa_m = \left.\frac{d^m K}{dt^m}\right|_{t=0}$

Factorial cumulants

+ integrated genuine correlation functions
  $\Rightarrow$ easier to interpret,
– not directly connected with statistical mechanics and rarely measured.

Generating func: $G(z) = \ln\left[\sum_{n=0}^{\infty} P(n) z^{n}\right]$

Factorial cumulants: $\hat{C}_m = \left.\frac{d^m G}{dz^m}\right|_{z=1}$

- Cumulants can be calculated from factorial cumulants:

$$\kappa_n = \sum_{k=1}^{n} S(n,k)\hat{C}_k \,,$$

where $S(n,k)$ is the Stirling number of the second kind.

B. Friman and K. Redlich, [arXiv:2205.07332 [nucl-th]].

- For example,

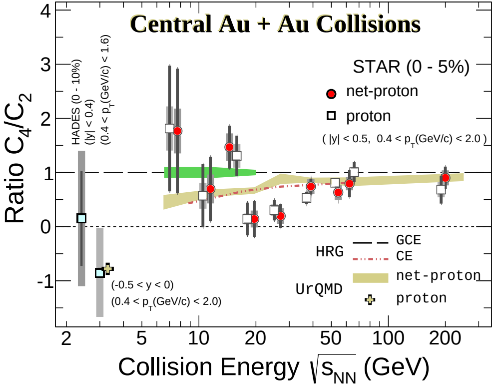
$\kappa_1 = \langle N \rangle = \hat{C}_1,$
$\kappa_2 = \langle N \rangle + \hat{C}_2,$
$\kappa_3 = \langle N \rangle + 3\hat{C}_2 + \hat{C}_3,$
$\kappa_4 = \langle N \rangle + 7\hat{C}_2 + 6\hat{C}_3 + \hat{C}_4.$

Poisson baseline

- Poisson distribution (no correlations):
  - cumulants $\kappa_m = \langle N \rangle$
    $\Rightarrow \frac{\kappa_m}{\kappa_n} = 1$,
  - factorial cumulants: $\hat{C}_m = 0,\ m > 1$
    $(\hat{C}_1 = \langle N \rangle)$.
- Skellam distribution - distribution of the difference between two Poissonian variables
  - baseline for, e.g., net-proton number, $(n_p - n_{\overline{p}})$.

STAR and HADES results

- κ_4/κ_2 (denoted by STAR as C_4/C_2) depends non-monotonically on energy.
- Possible signature of the critical phenomena?



[STAR], Phys. Rev. Lett. 128, no.20, 202303 (2022)

[HADES], Phys. Rev. C 102, no.2, 024914 (2020)

Effects contributing to fluctuations

- Fluctuations may reflect the desired phase transition
- ... but they may be due to other usual effects (background):
  - small fluctuations of the impact parameter (and thus the number of wounded nucleons)

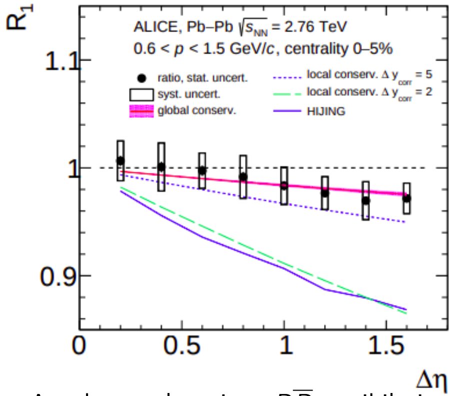
    V. Skokov, B. Friman and K. Redlich, Phys. Rev. C 88, 034911 (2013)
  - **global baryon number conservation**
    Baryon number conservation modifies cumulants.

    A. Bzdak, V. Koch and V. Skokov, Phys. Rev. C 87, no.1, 014901 (2013)
    V. Vovchenko, V. Koch and C. Shen, Phys. Rev. C 105, no.1, 014904 (2022)
    P. Braun-Munzinger, B. Friman, K. Redlich, A. Rustamov and J. Stachel, Nucl. Phys. A 1008, 122141 (2021)
- It's essential to know the "background" in order to extract the "signal".

ALICE results

- Global baryon number conservation is favored over local conservation and leads to long-range correlations.

  [ALICE], Phys. Lett. B 807, 135564 (2020)



$$R_1 = \frac{\kappa_2(n_p - n_{\bar{p}})}{\langle n_p + n_{\bar{p}} \rangle}$$

- normalized 2nd net-proton cumulant

- Another explanation: $B\overline{B}$ annihilation + local baryon conservation

O. Savchuk, V. Vovchenko, V. Koch, J. Steinheimer and H. Stoecker, PLB 827, 136983 (2022)

- It's necessary to study other cumulants and factorial cumulants.

Our results

Mixed factorial cumulants from global baryon conservation

MB and A. Bzdak, Phys. Rev. C 102, no.6, 064908 (2020)

Probability distribution

The probability distribution of observing n_p protons and $\bar{n}_p$ antiprotons is:

$$P(n_p, \bar{n}_p) = A \sum_{N_b = n_p}^{\infty} \sum_{\bar{N}_b = \bar{n}_p}^{\infty} \delta_{N_b - \bar{N}_b, B} \left[\frac{\langle N_b \rangle^{N_b}}{N_b!} e^{-\langle N_b \rangle} \right] \left[\frac{\langle \bar{N}_b \rangle^{\bar{N}_b}}{\bar{N}_b!} e^{-\langle \bar{N}_b \rangle} \right]$$
$$\times \left[\frac{N_b!}{n_p!(N_b - n_p)!} p^{n_p} (1-p)^{N_b - n_p} \right] \left[\frac{\bar{N}_b!}{\bar{n}_p!(\bar{N}_b - \bar{n}_p)!} \bar{p}^{\bar{n}_p} (1-\bar{p})^{\bar{N}_b - \bar{n}_p} \right],$$

where
N_b, $\bar{N}_b$ - # baryons and antibaryons,
B - *conserved* baryon number,
A - normalization constant,

baryon number conservation,
baryons, antibaryons number $\sim$ Poisson (no correlations),
binomial acceptance.

Generating function: $G(x, \bar{x}) = \ln \left[\sum_{n_p=0}^{\infty} \sum_{\bar{n}_p=0}^{\infty} x^{n_p} \bar{x}^{\bar{n}_p} P(n_p, \bar{n}_p) \right]$.
Mixed factorial cumulants: $\hat{C}^{(n,m)} = \frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial \bar{x}^m} G(x, \bar{x}) \Big|_{x=\bar{x}=1}$
for n protons and m antiprotons.

Mixed factorial cumulants can verify global conservation

- Mixed proton-antiproton factorial cumulants can be measured.
- They carry more information than net-proton cumulants.
- $\hat{C}^{(n,m)}$ - n-proton and m-antiproton factorial cumulant from global baryon number conservation.

$$\hat{C}^{(1,0)} = p\langle N_b\rangle_c$$

$$\hat{C}^{(2,0)} = -p^2\left(\langle N_b\rangle_c + \Delta\right)$$
$$\boldsymbol{\hat{C}^{(1,1)} = -p\bar{p}\Delta}$$

$$\hat{C}^{(3,0)} = p^3\left[2!\left(\langle N_b\rangle_c + \Delta + \tfrac{1}{2}\gamma\right)\right]$$
$$\hat{C}^{(2,1)} = p^2\bar{p}\,\gamma$$

$$\hat{C}^{(4,0)} = -p^4\left[3!\left(\langle N_b\rangle_c + \Delta + \tfrac{1}{2}\gamma\right) + \beta\right]$$
$$\hat{C}^{(3,1)} = -p^3\bar{p}\beta$$
$$\hat{C}^{(2,2)} = -p^2\bar{p}^2\left(\beta - \gamma\right)$$

Notation:
$p(\bar{p})$ - the probability that a (anti)baryon is inside the acceptance and measured as a (anti)proton,
$\langle N_b\rangle(\langle\bar{N}_b\rangle)$ - mean (anti)baryon number w/o baryon conservation,
$\langle N_b\rangle_c(\langle\bar{N}_b\rangle_c)$ - mean (anti)baryon number with baryon conservation,
$z = \sqrt{\langle N_b\rangle\langle\bar{N}_b\rangle}$, $z_c = \sqrt{\langle N_b\rangle_c\langle\bar{N}_b\rangle_c}$,
$\langle N\rangle_c = \langle N_b\rangle_c + \langle\bar{N}_b\rangle_c$,
$\Delta = z_c^2 - z^2$,
$\gamma = z_c^2 + \Delta\langle N\rangle_c$,
$\beta = \gamma(\langle N\rangle_c + 2) + 2\Delta^2$.

- if no correlations: $\hat{C}^{(n,m)} = 0$ (except $\hat{C}^{(1,0)}$, $\hat{C}^{(0,1)}$)

Baryon factorial cumulants and cumulants from global baryon conservation with short-range correlations

MB and A. Bzdak, Phys. Rev. C 106, no. 2, 024904 (2022)

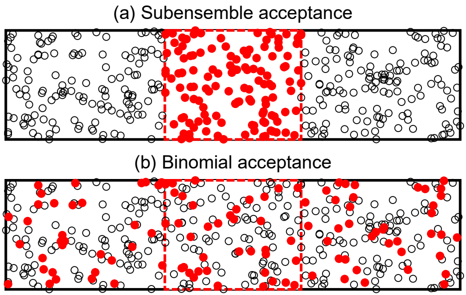
MB and A. Bzdak, [arXiv:2210.15394 [hep-ph]]

SAM and net-baryon cumulants

V. Vovchenko, O. Savchuk, R.V. Poberezhnyuk, M.I. Gorenstein, V. Koch,

Phys. Lett. B 811, 135868 (2020)

- Subensemble acceptance method (SAM) (short-range correlations).
- Statistical mechanics, thermodynamic limit.
- They derived net-baryon cumulants with baryon number conservation and short-range correlations in terms of cumulants without baryon conservation.



Short- and long-range correlations

A. Bzdak, V. Koch and N. Strodthoff, Phys. Rev. C 95, no.5, 054906 (2017)

Short-range correlations:

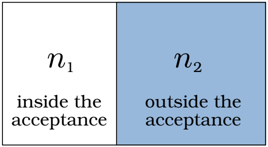
- correlations are local in rapidity and depend only on the relative distances.
- $\hat{C}_m \sim \Delta y \sim \langle N \rangle$
- $\Rightarrow \kappa_m \sim \langle N \rangle$
- $\Rightarrow \frac{\kappa_m}{\kappa_n} \sim const$

Long-range correlations:

- correlations are constant over the whole rapidity region.
- $\hat{C}_m \sim (\Delta y)^m \sim \langle N \rangle^m$
- $\Rightarrow \kappa_m \sim \sum_{k=1}^{m} \alpha_k \langle N \rangle^k$
- $\Rightarrow \frac{\kappa_m}{\kappa_n}$ - ratio of polynomials

Problem formulation

- The system is divided into two subsystems.



- Only baryons, no antibaryons

  (applies to low energies where the critical point is expected).

- Probability distribution: $P_B(n_1, n_2) = A\ P_1(n_1)P_2(n_2)\, \underbrace{\delta_{n_1+n_2,B}}_{\text{B conservation}}$,

  n_1 baryons in the first subsystem (inside the acceptance),
  n_2 baryons in the second one (outside the acceptance),
  A - normalization const.

- P_1, P_2 include only short-range correlations: $\hat{C}_k^{(i)} = \langle n_i \rangle \alpha_k$, $i = 1, 2$,
  α_k - k-particle short-range correlation strength

- Inside the acceptance: $P_B(n_1) = \sum_{n_2} P_B(n_1, n_2)$.

Calculation

- The factorial cumulant generating function in the 1st subsystem with baryon number conservation:
  $G_{(1,B)}(z) = \ln\left[\sum_{n_1} P_B(n_1) z^{n_1}\right]$,
  $G_{(1,B)}(z) = \ln\left[\frac{A}{B!}\, \frac{d^B}{dx^B} \exp\left(\sum_{k=1}^{\infty} \frac{(xz-1)^k \hat{C}_k^{(1)} + (x-1)^k \hat{C}_k^{(2)}}{k!}\right)\Big|_{x=0}\right]$.
- The factorial cumulants in the acceptance can be calculated:

$$\hat{C}_k^{(1,B)} = \frac{d^k}{dz^k} G_{(1,B)}(z)\Big|_{z=1} .$$

- Cumulants are calculated from factorial cumulants.
- Details of calculations in our papers.

Approximate factorial cumulants

- Approximated results for small α_k and in the limit of large B:
  $\hat{C}_1^{(1,B)} = f\,B,$
  $\hat{C}_2^{(1,B)} \approx f\,B\left[-f + \bar{f}\alpha_2\right],$
  $\hat{C}_3^{(1,B)} \approx f\,B\left[2f^2 - 6\bar{f}f\alpha_2 + \bar{f}(1-2f)\alpha_3\right],$
  $\hat{C}_4^{(1,B)} \approx f\,B\left[-3!f^3 + 36\bar{f}f^2\alpha_2 - 12\bar{f}f(1-2f)\alpha_3 + \bar{f}(1-3\bar{f}f)\alpha_4\right]$

  f - a fraction of particles in the acceptance, $\bar{f} = 1 - f$
- **For large B, nth factorial cumulant is not influenced by k-particle short-range correlations with $k > n$.**
  For example, in $\hat{C}_3^{(1,B)}$ only 2- and 3-particle short-range correlations are significant (higher-order correlations suppressed).
- Short-range factorial cumulants: $\hat{C}_k^{(i)} = \langle n_i \rangle \alpha_k$, $i = 1, 2$

  α_k - k-particle short-range correlation strength

Cumulants with B conservation and short-range correlations

They can be calculated from the cumulants without baryon conservation.

Cumulants as the power series in terms of B

$$\kappa_n^{(1,B)} \approx \underbrace{\kappa_n^{(1,B,\mathrm{LO})}}_{\substack{\propto B^1 \\ \text{thermodynamic limit}}} + \underbrace{\kappa_n^{(1,B,\mathrm{NLO})}}_{\propto B^0} + \underbrace{\dots}_{O(B^{-1})}$$

$$\kappa_1^{(1,B)} = fB = f\kappa_1^{(G)}$$

$$\kappa_2^{(1,B,\mathrm{LO})} = \bar{f} f \kappa_2^{(G)}$$

$$\kappa_2^{(1,B,\mathrm{NLO})} = \frac{1}{2}\bar{f} f \frac{(\kappa_3^{(G)})^2 - \kappa_2^{(G)}\kappa_4^{(G)}}{(\kappa_2^{(G)})^2}$$

$$\kappa_3^{(1,B,\mathrm{LO})} = \bar{f} f (1-2f)\kappa_3^{(G)}$$

$$\kappa_3^{(1,B,\mathrm{NLO})} = \frac{1}{2} f \bar{f} (1-2f) \frac{\kappa_3^{(G)}\kappa_4^{(G)} - \kappa_2^{(G)}\kappa_5^{(G)}}{(\kappa_2^{(G)})^2}$$

- LO calculated in a different way reproduces net-baryon cumulants from

  V. Vovchenko, O. Savchuk, R.V. Poberezhnyuk, M.I. Gorenstein, V. Koch, PLB 811, 135868 (2020)
- NLO is new.

$\kappa_n^{(1,B)}$ - cumulants in the subsystem with the baryon conservation and short-range correlations
$\kappa_n^{(G)}$ - short-range cumulants in the whole system without baryon conservation

f - a fraction of particles in the acceptance, $\bar{f} = 1 - f$

Cumulants with B conservation and short-range correlations

$$\kappa_n^{(1,B)} \approx \underbrace{\kappa_n^{(1,B,\mathrm{LO})}}_{\substack{\propto B^1 \\ \text{thermodynamic limit}}} + \underbrace{\kappa_n^{(1,B,\mathrm{NLO})}}_{\propto B^0} + \underbrace{\ldots}_{O(B^{-1})}$$

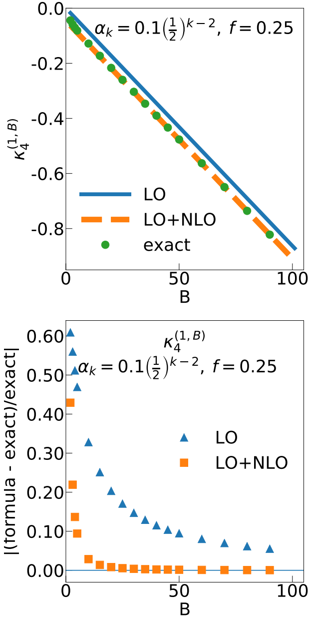
$$\kappa_4^{(1,B,\mathrm{LO})} = f\bar{f}\left[\kappa_4^{(G)} - 3f\bar{f}\left(\kappa_4^{(G)} + (\kappa_3^{(G)})^2/\kappa_2^{(G)}\right)\right]$$

$$\kappa_4^{(1,B,\mathrm{NLO})} = \frac{1}{2}f\bar{f}\left\{\frac{\kappa_3^{(G)}\kappa_5^{(G)} - \kappa_2^{(G)}\kappa_6^{(G)}}{(\kappa_2^{(G)})^2} + 3f\bar{f}\left[\frac{2(\kappa_3^{(G)})^4 - 5\kappa_2^{(G)}(\kappa_3^{(G)})^2\kappa_4^{(G)} + (\kappa_2^{(G)})^2\kappa_3^{(G)}\kappa_5^{(G)}}{(\kappa_2^{(G)})^4} + \frac{(\kappa_4^{(G)})^2 + \kappa_2^{(G)}\kappa_6^{(G)}}{(\kappa_2^{(G)})^2}\right]\right\}$$

$\kappa_n^{(1,B)}$ - cumulants in the subsystem with the baryon conservation and short-range correlations,
$\kappa_n^{(G)}$ - short-range cumulants in the whole system without baryon conservation,
f - a fraction of particles in the acceptance, $\bar{f} = 1 - f$.

- LO calculated in a different way reproduces net-baryon cumulants from
  V. Vovchenko, O. Savchuk, R.V. Poberezhnyuk, M.I. Gorenstein, V. Koch, PLB 811, 135868 (2020).
- NLO is new.

Example



- exact - a straightforward differentiation of the factorial cumulant generating function,
- α_k - k-particle short-range correlation strength, $\alpha_k = 0.1\left(\frac{1}{2}\right)^{k-2}$, $k = 2...6$, $\alpha_1 = 1$,
- f - a fraction of particles in the acceptance.

- NLO improves the results.

Ongoing study

- The measured $dN/dy = \rho(y)$ is $\langle dN/dy \rangle$.
- Low energies (< 10 GeV) it's $\approx$ Gaussian.
- Its width can fluctuate e-by-e due to baryon stopping fluctuations.

A. Bzdak, D. Teaney, PRC **87**, no.2, 024906 (2013)

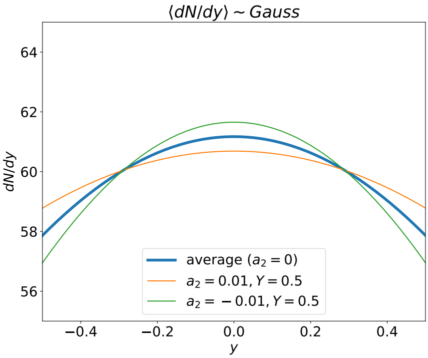
A. Bzdak, P. Bozek, PRC **93**, no.2, 024903 (2016)

$$\rho(y) = \langle \rho(y) \rangle [1 + a_2 T_2(y/Y)]$$

Y - the scale of long-range rapidity fluctuations

$T_2()$ - second scaled Legendre polynomial

- This can affect cumulants.
- Work in progress



Summary

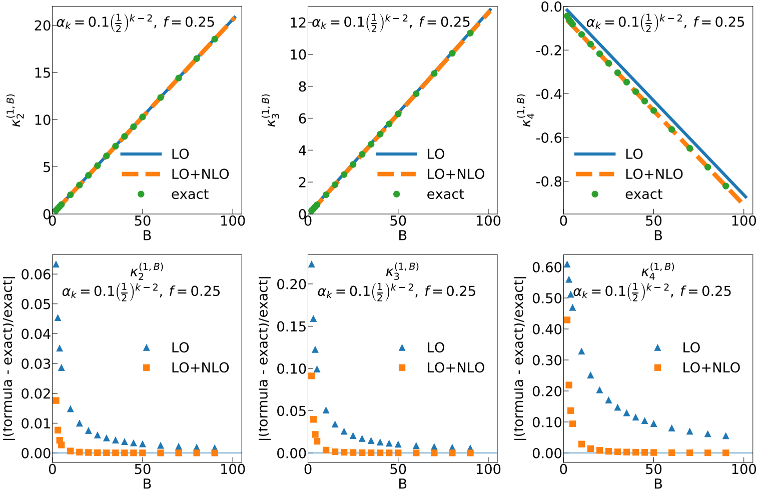
- In order to search for QCD phase transition, the measured correlations (cumulants and factorial cumulants) of conserved charges are studied.
- Different effects not related to phase transition or critical phenomena are also responsible for fluctuations.
- The mathematical model for calculating cumulants from global baryon conservation and short-range correlations was presented.
- Mixed factorial cumulants from baryon conservation can be helpful to understand ALICE data.
- Cumulants with baryon conservation and short-range correlations can be calculated having the cumulants without baryon conservation.
- NLO correction to baryon cumulants improves the results.
- Future plans: study longitudinal correlations due to fireball shape fluctuations.

Thank you

Backup

Example

- exact - a straightforward differentiation of the factorial cumulant gen. func.
- α_k - k-particle short-range correlation strength ($\alpha_k = 0.1\left(\frac{1}{2}\right)^{k-2}$, $k = 2...6$, $\alpha_1 = 1$)
- f - a fraction of particles in the acceptance



- NLO improves the results.