Entanglement entropy and proton's structure

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Motivation

Bounds and properties of EE may provide some new insight on behavior of pdfs

Links to other areas (thermodynamics, gravity, quantum information, conformal field theory)

Interesting in context of parton saturation and thermalization problem of Quark Gluon Plasma

Various approaches to entropy in the low x limit: entropy of gluon density, thermodynamic entropy
momentum space entanglement, coordinate space entanglement, Wehrl entropy,...

Based on:

Based on

Eur.Phys.J.C 82 (2022) 2, 111
M. Hentschinski, K. Kutak

Eur.Phys.J.C 82 (2022) 12, 1147
M. Hentschinski, K.Kutak, R. Straka

Arxiv:2305.03069
H. Hentschinski, D. Kharzeev. K. Kutak, Z. Tu

Arxiv: 1103.3654.v1 and v2
K. Kutak

Boltzman and von Neuman entropy formulas – reminder

The entropy S of macrostate is given by the log of number W of distinct microstates that compose it

$$S = -\sum_{i=1}^{W} p(i) \ln p(i)$$ Gibbs entropy

For uniform distribution $p(i) = \frac{1}{W}$ the entropy is maximal

Boltzmann entropy

$$S = \ln W$$

Since partons are introduced as the microscopic constituents that compose the macroscopic state of the proton, it is natural to evaluate the corresponding entropy or entropy corresponding to parton density.

K. Kutak '11, Peschanski'12
A. Kovner, M. Lublinsky '15
D. Kharzeev, E. Levin '17,...

But proton as a whole is a pure state and the von Neuman entropy is 0. Can one get any nontrivial result?

For pure state (one state) density matrix is:

$$\rho = |\psi\rangle\langle\psi|$$

$$S_{VN} = -Tr[\rho \ln \rho] = -1 \ln 1 = 0$$

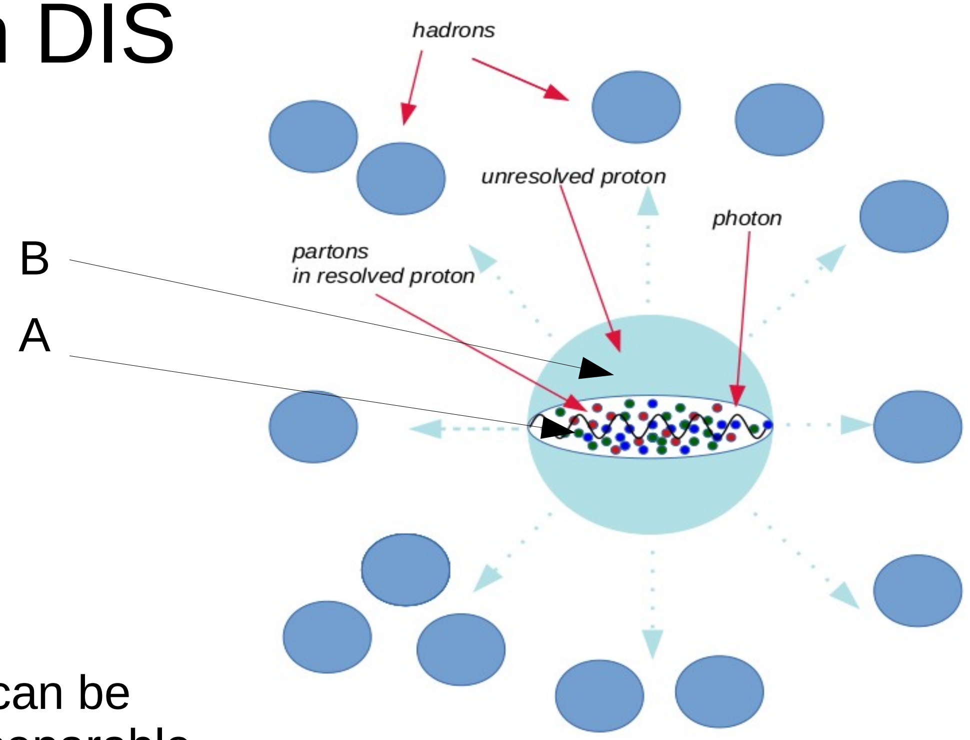
For mixed state i.e. classical statistical mixture

$$\rho = \sum p(i) |\psi_i\rangle\langle\psi_i|$$

$$S_{VN} \neq 0$$

Kharzeev, Levin '17

Entanglement entropy in DIS



The composite system is described by

$|\Psi_{AB}\rangle$ in $A \cap B$

proton's rest frame

$\mathcal{H}_B$ of dimension n_B.

$\mathcal{H}_A$ of dimension n_A

Kharzeev, Levin '17

entangled
if the product can not be expressed as separable product state

$$|\Psi_{AB}\rangle = \sum_{i,j} c_{ij} |\varphi_i^A\rangle \otimes |\varphi_j^B\rangle$$

separable
if the product can be expressed as separable product state

$$|\Psi_{AB}\rangle = |\varphi^A\rangle \otimes |\varphi^B\rangle$$

Schmidt decomposition

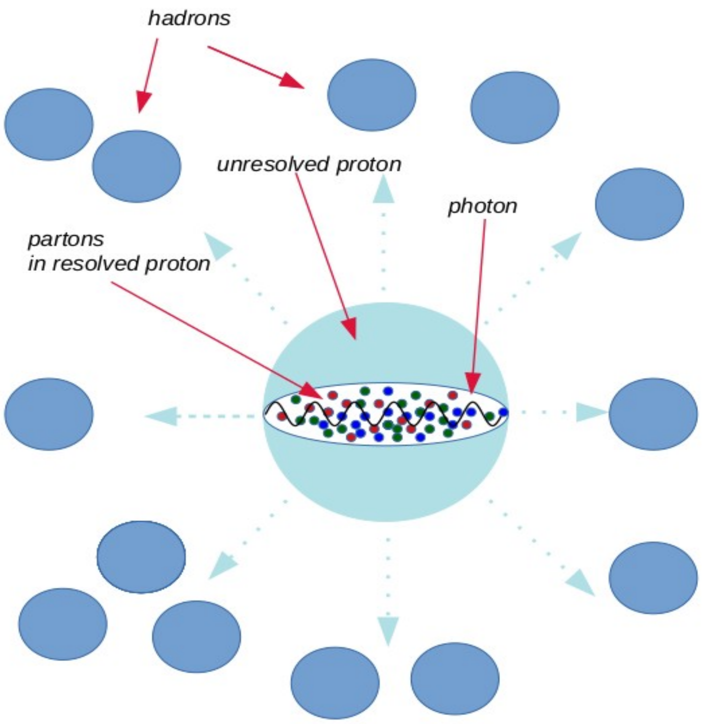
$$|\Psi_{AB}\rangle = \sum \alpha_n |\Psi_n^A\rangle |\Psi_n^B\rangle$$

orthonormal states belonging to A

orthonormal states bolonging to B

related to matrix C

Entanglement entropy in DIS



$$|\Psi_{AB}\rangle = \sum_n \alpha_n |\Psi_n^A\rangle|\Psi_n^B\rangle$$

$$\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$$

$$\rho_A = \mathrm{tr}_B \rho_{AB} = \sum_n \alpha_n^2 |\Psi_n^A\rangle\langle\Psi_n^A|$$

The density matrix of the mixed state probed in region A

$\alpha_n^2 \equiv p_n$ probability of state with n partons

Kharzeev, Levin '17

$$S = -\sum_n p_n \ln p_n$$

entropy results from the entanglement between the regions A and B, and can thus be interpreted as the entanglement entropy. Entropy of region A is the same as entropy in region B.

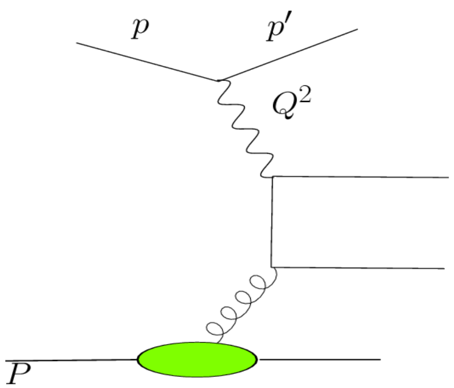
Proton structure function and dipole cross section

$$F_2(x,Q^2) = \frac{Q^2}{4\pi^2}\alpha_s \sum_q e_q^2 \int d^2k\, \mathcal{F}(x,k^2)\left(S_L(k^2,Q^2,m_q^2) + S_T(k^2,Q^2,m_q^2)\right)$$

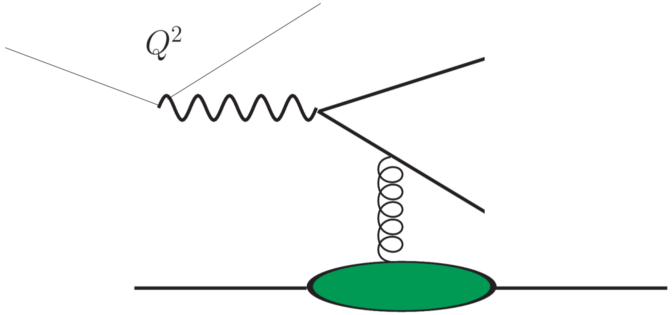
dipole gluon density

impact factors ~ hard coefficients

In the kt factorization



In the dipole formalism

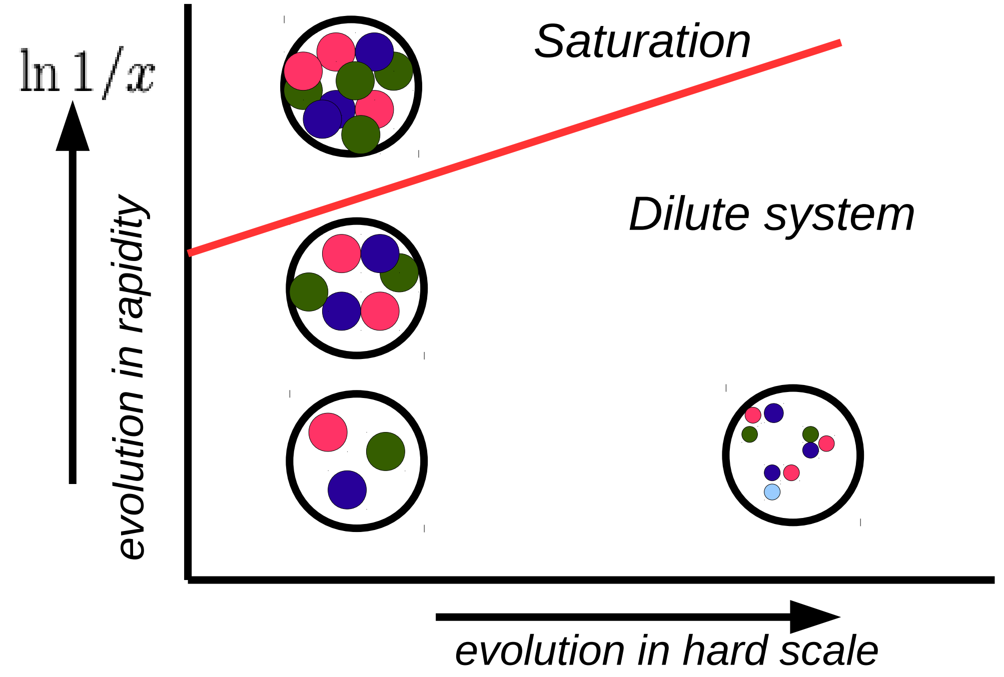


$$F_2(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{em}} \int d^2b \int_0^1 dz \int d^2r \left(|\psi_L(z,r)|^2 + |\psi_T(z,r)|^2\right) N(x,r,b)$$

wave function

dipole amplitude

Gluons at high energies



Saturation – state where number of gluons stops growing due to high occupation number. Way to fulfill unitarity requirements in high energy limit of QCD.

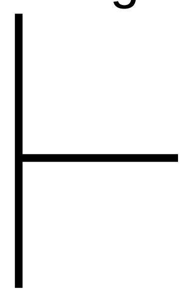
L.V. Gribov, E.M. Levin, M.G. Ryskin
Phys.Rept. 100 (1983) 1-150

Larry D. McLerran, Raju Venugopalan
Phys.Rev. D49 (1994) 3352-3355

On microscopic level it means that
gluon apart splitting recombine

Linear evolution
Equation
BFKL

splitting

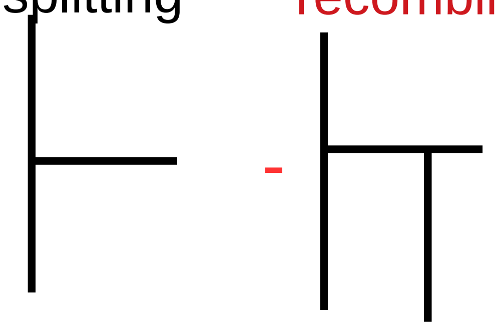


Nonlinear evolution equations

BK, JIMWLK
Balitcky-Kovchegov,

Jailian-Marian,Iancu
McLerran,Weigert,Leonidov,Kovner

splitting recombination



Bartels,
Wusthoff ‘93

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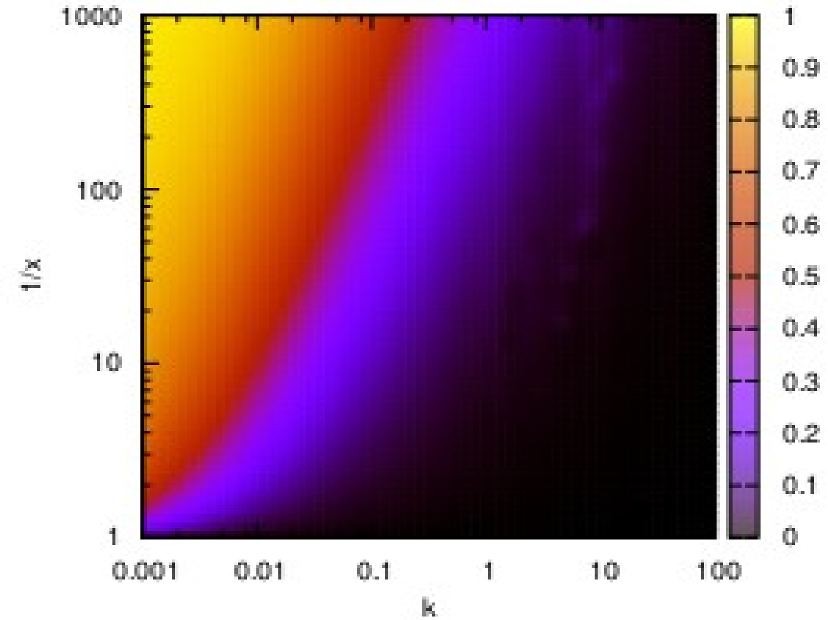

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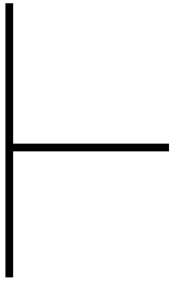

splitting recombination

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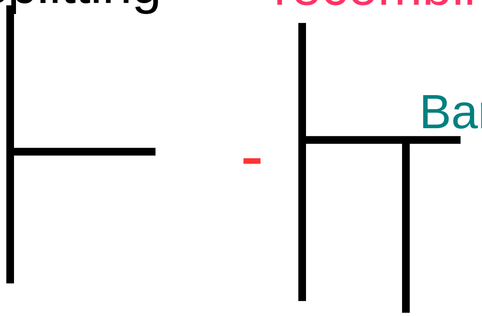
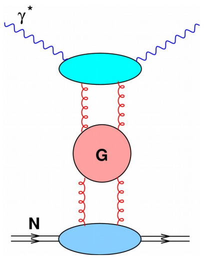


Bartels, Wusthoff'93

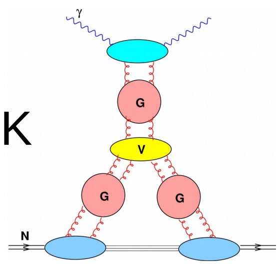
Momentum space vs coordinate space

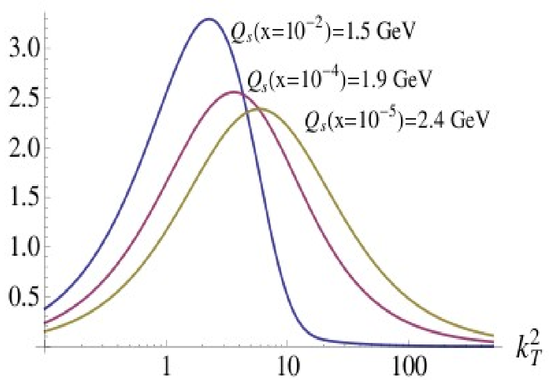
momentum space - Bjorken frame

BFKL

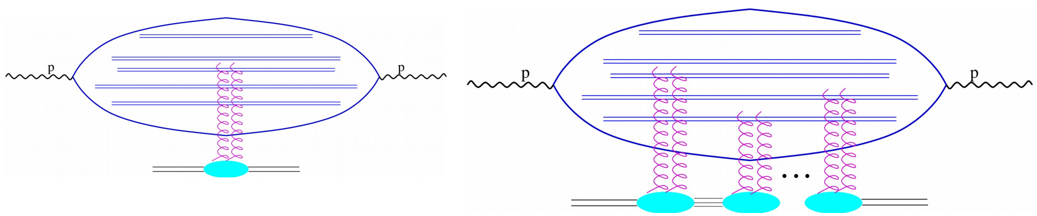


BK

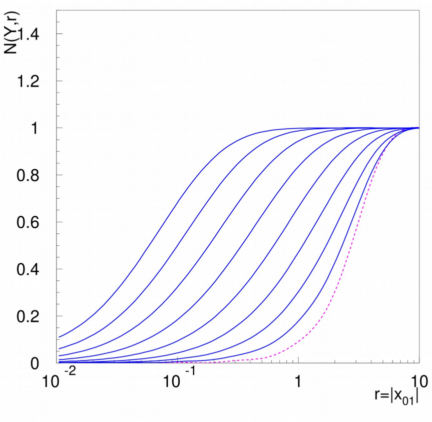




position space - Mueller frame



gluon ~ color dipole



from *A. Stasto Acta Phys.Polon. B35 (2004) 3069-3102*

$$\mathcal{F}(x,k) = \mathcal{F} + K_{m\,s} \otimes \mathcal{F}(x,k) - \frac{1}{R^2} TPV \otimes \mathcal{F}(x,k)^2 \qquad N(x,r,b) = N_0 + K_{p\,s} \otimes \left(N(x,r,b) - N(x,r,b)^2\right)$$

dipole unintegrated gluon density

Evolved with BK dipole amplitude – expectation value of product of Wilson lines in fundamental representation

related by Fourier transform

The dipole cross section and integrated gluon

$$\sigma(x,r) = \frac{4\pi^2}{N_c}\int \frac{dk^2}{k^2}(1 - J_0(kr))\mathcal{F}(x,k^2)$$

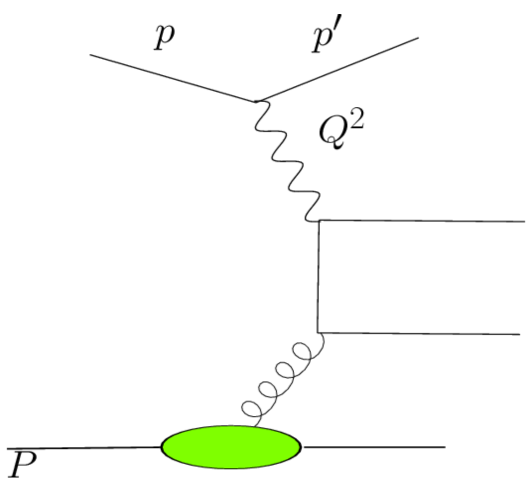
$$\sigma(x,r) \approx \frac{4\pi^2}{N_c}\int \frac{dk^2}{k^2}\left(1 - \left(1 - \frac{k^2r^2}{4}\right)\right)\mathcal{F}(x,k^2)$$

$$xg(x,Q^2) \equiv \int^{Q^2} dk^2 \mathcal{F}(x,k^2)$$

$$\sigma(x,r) \approx \frac{\pi^2}{N_c} r^2 xg(x,1/r^2)$$

$$\sigma(x,r) = \sigma_0 N(x,r)$$

$$N(x,r) \approx xg(x,1/r^2)$$



integrated gluon density

For fixed dipole size one has.

In the context of the scale dependent GBW model this approximation is viewed as linear approximation

$$N(x) = 1 - Z(x)$$

$$Z(x) \propto \sum_n P_n$$

generating function for dipoles

Partonic, dipole cascade

$$p_n = P_n$$

$$\frac{dP_n(Y)}{dY} = -\lambda n P_n(Y) + (n-1)\lambda P_{n-1}(Y)$$

set of partons is described by set of dipoles with fixed sizes ,Y is rapidity and is related to energy

Mueller 95, Lublinsky, Levin '03

depletion of the probability to find n dipoles due to the splitting into (n + 1) dipoles.

the growth due to the splitting of (n − 1) dipoles into n dipoles.

$$P_n(Y) = e^{-\lambda Y}\left(1 - e^{-\lambda Y}\right)^{n-1}$$

$$S = -\sum_n p_n \ln p_n$$

model of BFKL dipole cascade

$S(Y) \approx \lambda Y$ where $Y = \ln 1/x$

$$\langle n \rangle = \sum_n n P_n(Y) = \left(\frac{1}{x}\right)^{\lambda}$$

BFKL intercept = $4 \ln 2\, \bar{\alpha}_S$

Kharzeev, Levin '17

Assumption $\langle n \rangle \equiv xg(x)$

$$S(x) = \ln(xg(x))$$

The approach can be generalized to 3+1 d and one can account for hard scale dependence.

Density matrix in 1+1 D Nowak, Liu, Zahed '22

EE in DLL Nowak, Liu, Zahed '23

$$S(x, Q) = \ln(xg(x, Q))$$

Entropy formula - interpretation

At low x partonic microstates have equal probabilities

$$P_n(Y) = e^{-\lambda Y}\left(1 - e^{-\lambda Y}\right)^{n-1}$$

In this equipartitioned state the entropy is maximal – the partonic state at small x is maximally entangled.

In terms of information theory as Shanon entropy:
- equipartitioning in the maximally entangled state means that all “signals” with different number of partons are equally likely
- it is impossible to predict how many partons will be detected in a give event.
- structure function at small x should become universal for all hadrons.

From strict bounds on entanglement entropy (from conformal field theory) one can obtain that at low x (in conformal regime) one has

$$xg(x) \leq \text{const}\, x^{-1/3}$$

Kharzeev, Levin ‘17

Furthermore entropy of the final state hadrons can not be smaller than entropy of partons.

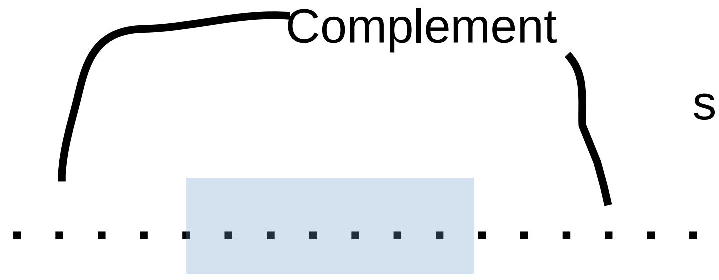
Comments

CFT result for EE

central charge

$$S = \frac{c}{3} \ln \frac{L}{\epsilon}$$

UV cutoff



spnis in 1 D

Region A of length L

Entanglement entropy obtained from CFT calculations as well as from gravity using Ryu-Takayanagi formula

Relation to Kharzeev-Levin formula

$$L = (mx)^{-1} \qquad \epsilon \equiv 1/m$$

Length of tube probed in DIS

Proton's Compton wave length

$$S = \ln \left(\frac{1}{x}\right)^{1/3}$$

$$S(x) = \ln(xg(x))$$

See also
Callan, Wilczek '94
Calabrese, Cardy '04

and lectures by Headrick

Studied also in the context of 2 D QCD

Liu, Nowak, Zahed, '22

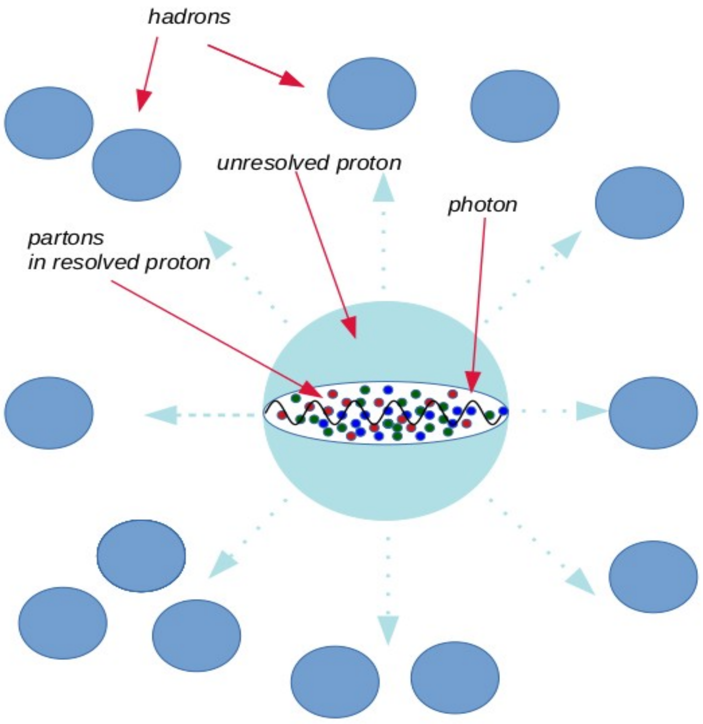
Casini, Huerta, Hosco '05

Entanglement entropy – calculation and measurement

For DIS at high energies, this entanglement entropy can be calculated using

$$S(x, Q^2) = \ln \left\langle n \left(\ln \frac{1}{x}, Q \right) \right\rangle$$

$$S_{hadron} = \sum P(N) \ln P(N)$$

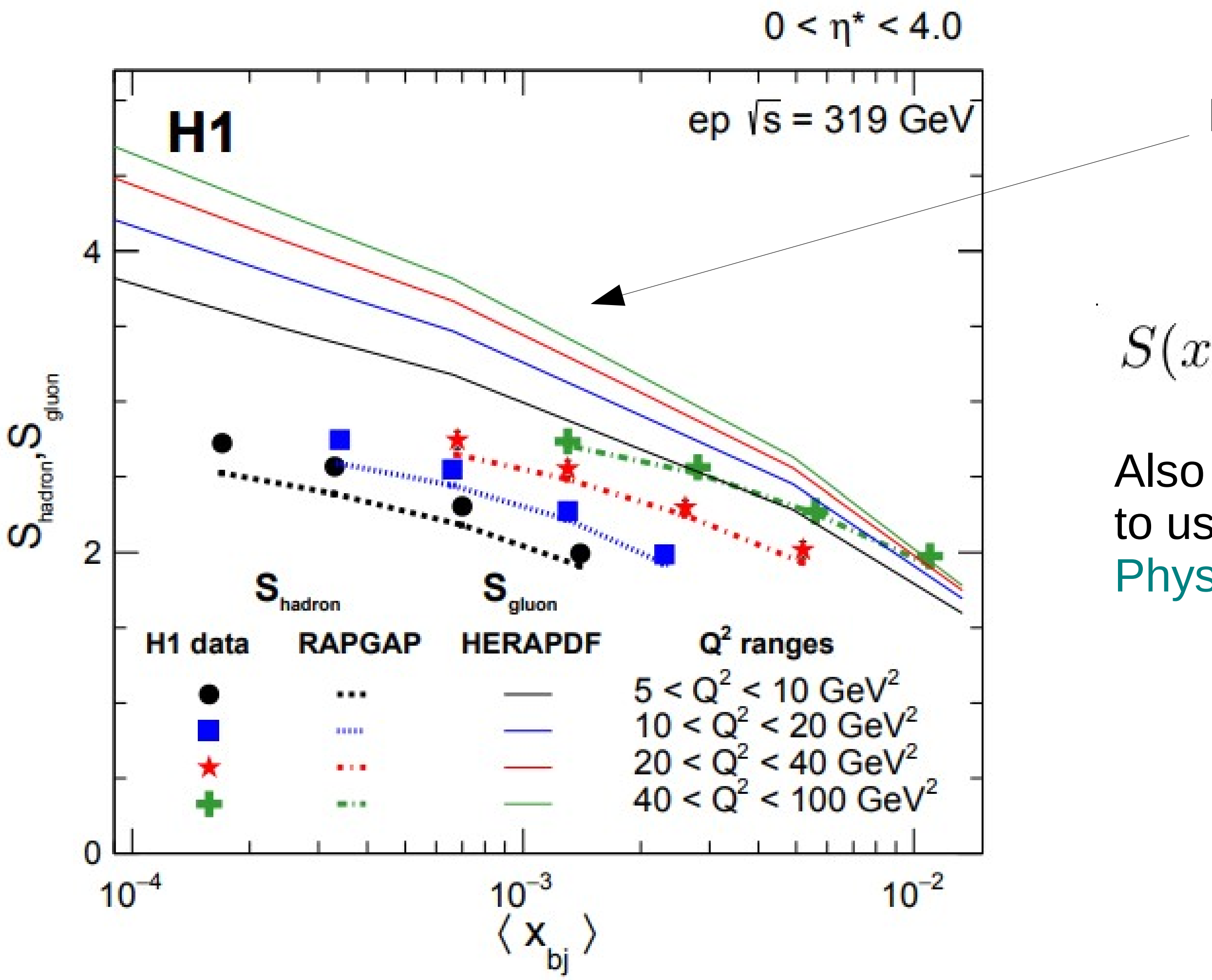


N number of measured hadrons

The charged particle multiplicity distribution measured in either the current fragmentation region or the target fragmentation region.

Fraction of events with charged hadron

Data and EE



$$S(x, Q) = \ln(xg(x, Q))$$

Also attempt by Kharzeev and Levin to use quarks instead of gluons
Phys. Rev. D 104, 031503 (2021)

H1

Eur.Phys.J.C 81 (2021) 3, 212

See also Z. Tu, D. Kharzeev, T. Ulrich ’20 for calculations of EE in p-p.

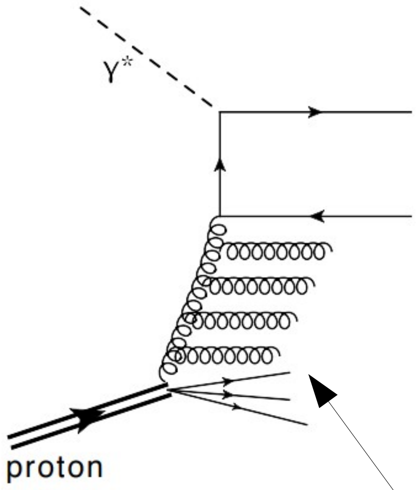
Extension of KL entropy formula

Hentschinski, Kutak ‘21

$$\left\langle n\left(\ln\frac{1}{x},Q\right)\right\rangle = xg(x,Q)+x\Sigma(x,Q)$$

To get the entropy of system of partons one needs to account for both quarks and gluons. One can view this as a higher order correction to KL formula. Furthermore it is impossible to isolate quarks from gluons therefore the compete entropy formula should receive contributions from quarks and gluons

Gluon and quark distribution



In the linear regime obeys BFKL equation. In our calculations we use NLO BFKL with kinematical improvements and running coupling. The gluon density has been fitted to F_2 data (exact kinematics was used)

Hentschinski, Sabio-Vera, Salas.
Phys.Rev.D 87 (2013) 7, 076005
Phys.Rev.Lett. 110 (2013) 4, 041601

We calculate the sea quarks distribution using

$$x\Sigma(x,Q) = P_{qg}(Q,\mathbf{k}) \otimes \mathcal{F}(x,\mathbf{k}^2)$$

$$xg(x,Q) = \int_0^{Q^2} d\mathbf{k}^2 \mathcal{F}(x,\mathbf{k}^2)$$

Transverse momentum dependent splitting function
Catani, Hautmann
Nucl.Phys. B427 (1994) 475-524

Other methods for resummation:
KMS (Kwiecinski, Martin, Stasto);
CCSS (Colferai, Ciafaloni, Staśto, Salam)

Gluon distribution

NLO BFKL with collinear resummation

$$\mathcal{F}\left(x,\boldsymbol{k}^2,Q\right)=\frac{1}{\boldsymbol{k}^2}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{d\gamma}{2\pi i}\ \hat{g}\left(x,\frac{Q^2}{Q_0^2},\gamma\right)\ \left(\frac{\boldsymbol{k}^2}{Q_0^2}\right)^{\gamma}$$

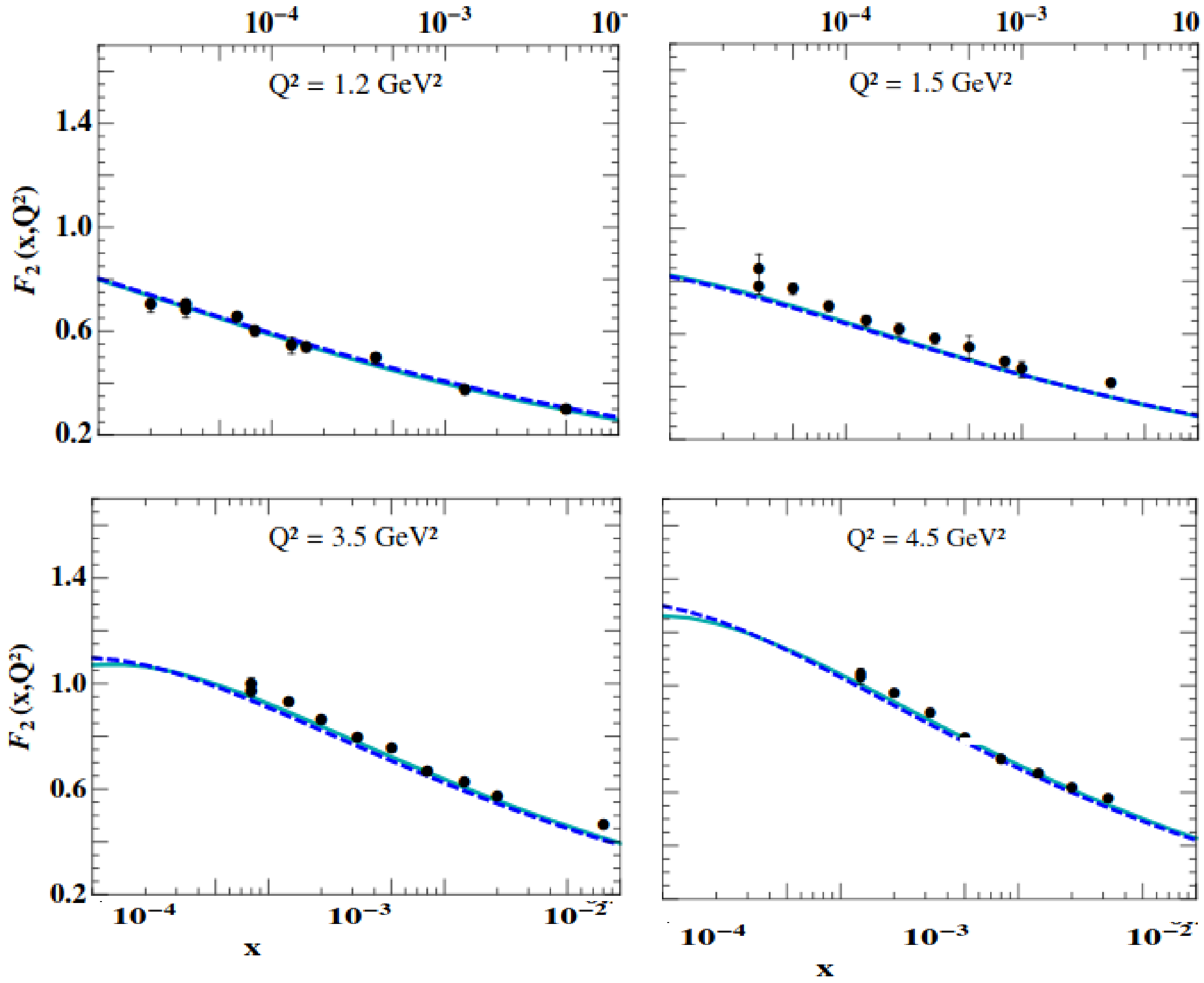
$$\hat{g}\left(x,\frac{Q^2}{Q_0^2}\gamma\right)=\frac{\mathcal{C}\cdot\Gamma(\delta-\gamma)}{\pi\Gamma(\delta)}\cdot\left(\frac{1}{x}\right)^{\chi(\gamma,Q,Q)}\left\{1+\frac{\bar{\alpha}_s^2\beta_0\chi_0\left(\gamma\right)}{8N_c}\log\left(\frac{1}{x}\right)\left[-\psi\left(\delta-\gamma\right)+\log\frac{Q^2}{Q_0^2}-\partial_\gamma\right]\right\}$$

the low x growth

Hentschinski, Sabio-Vera, Salas.
Phys.Rev.D 87 (2013) 7, 076005
Phys.Rev.Lett. 110 (2013) 4, 041601

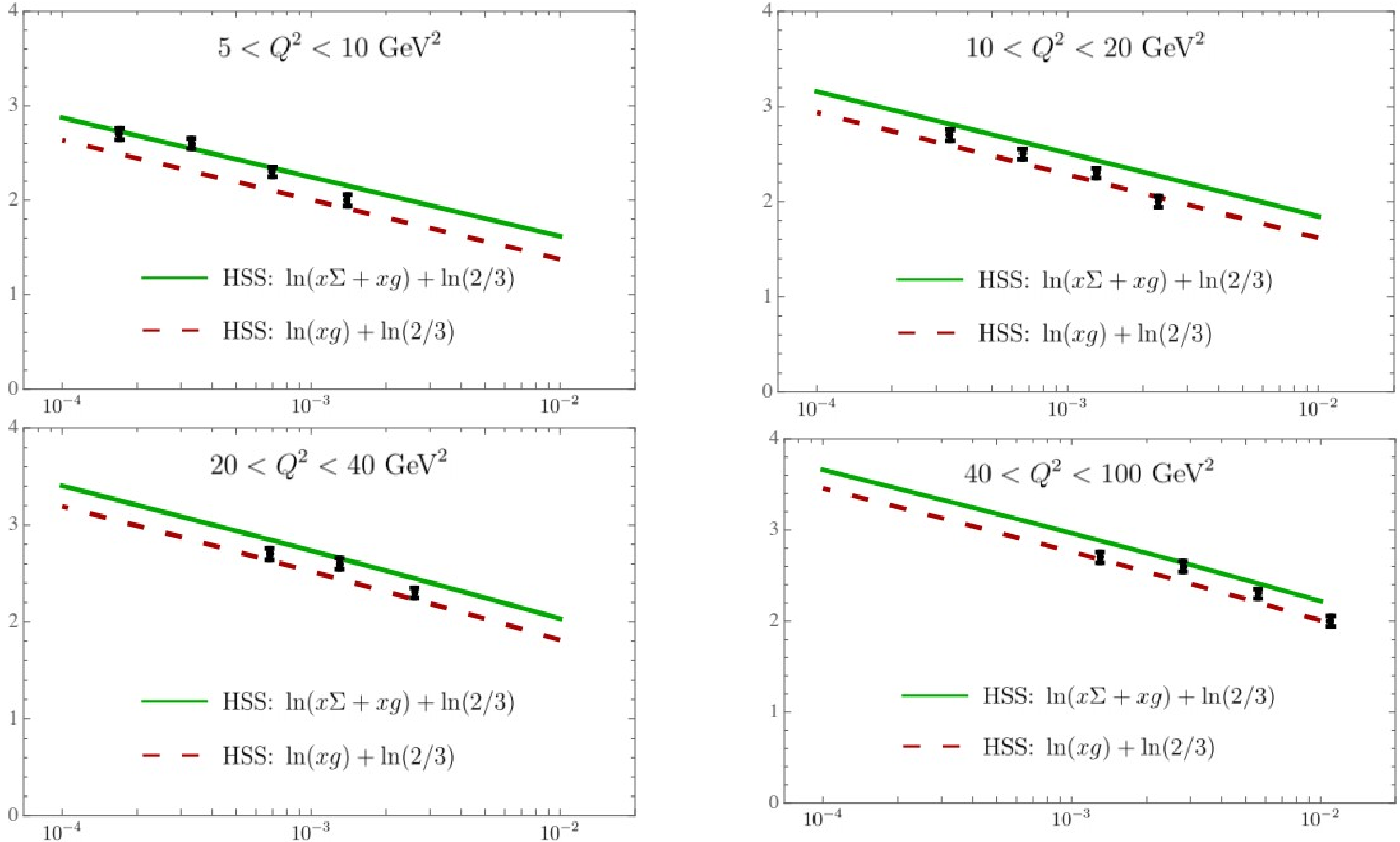
Proton structure function from HSS fit

F_2 data description



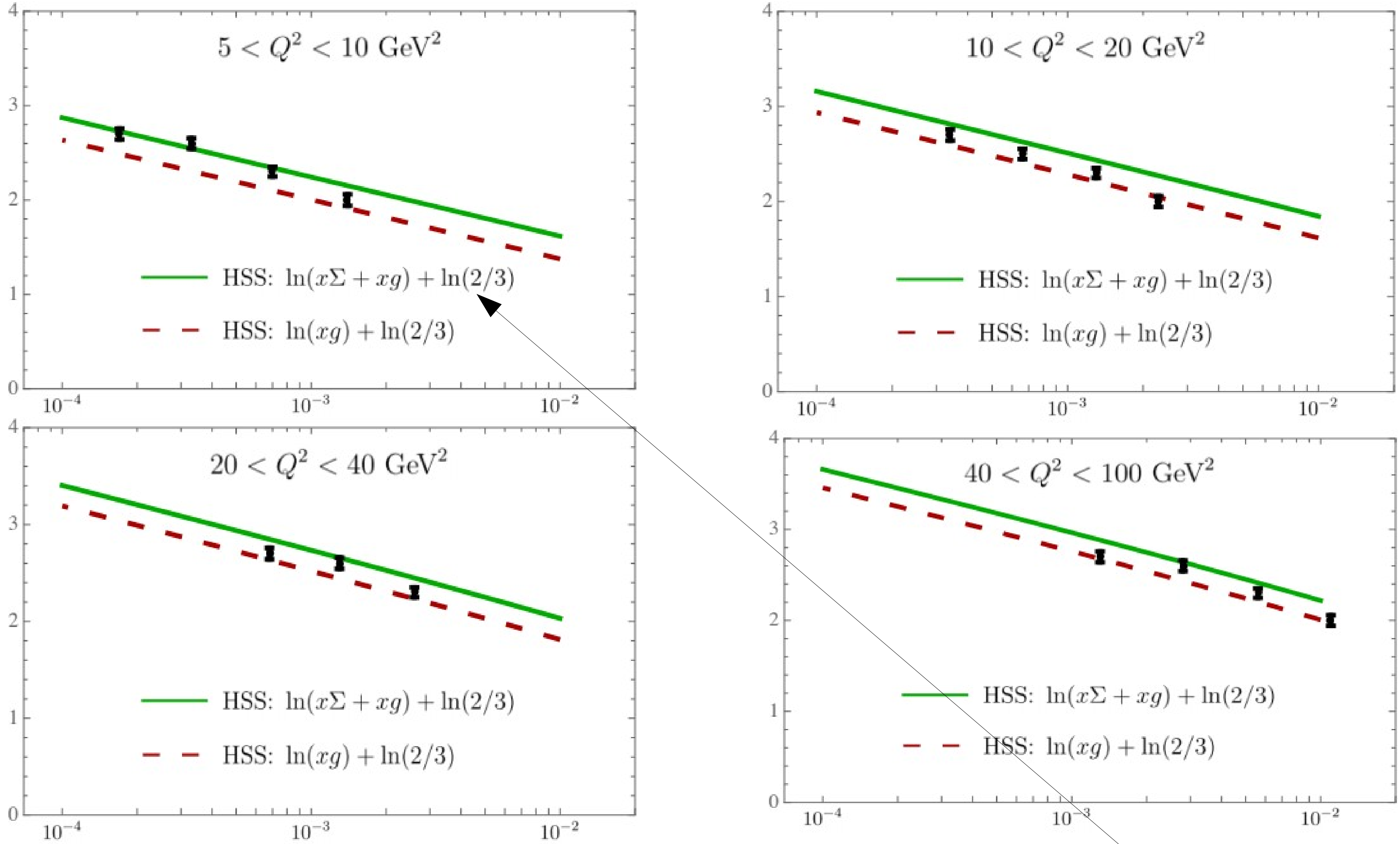
Eur.Phys.J.C 82 (2022) 2, 111 Hentschinski, Kutak

Results



Hint that the general idea works. Gluon dominates over quarks.
One has to also take into account that only charged hadrons were measured.

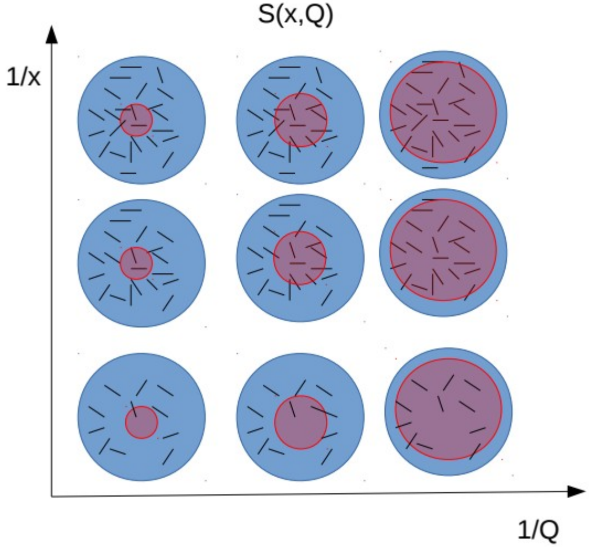
Results

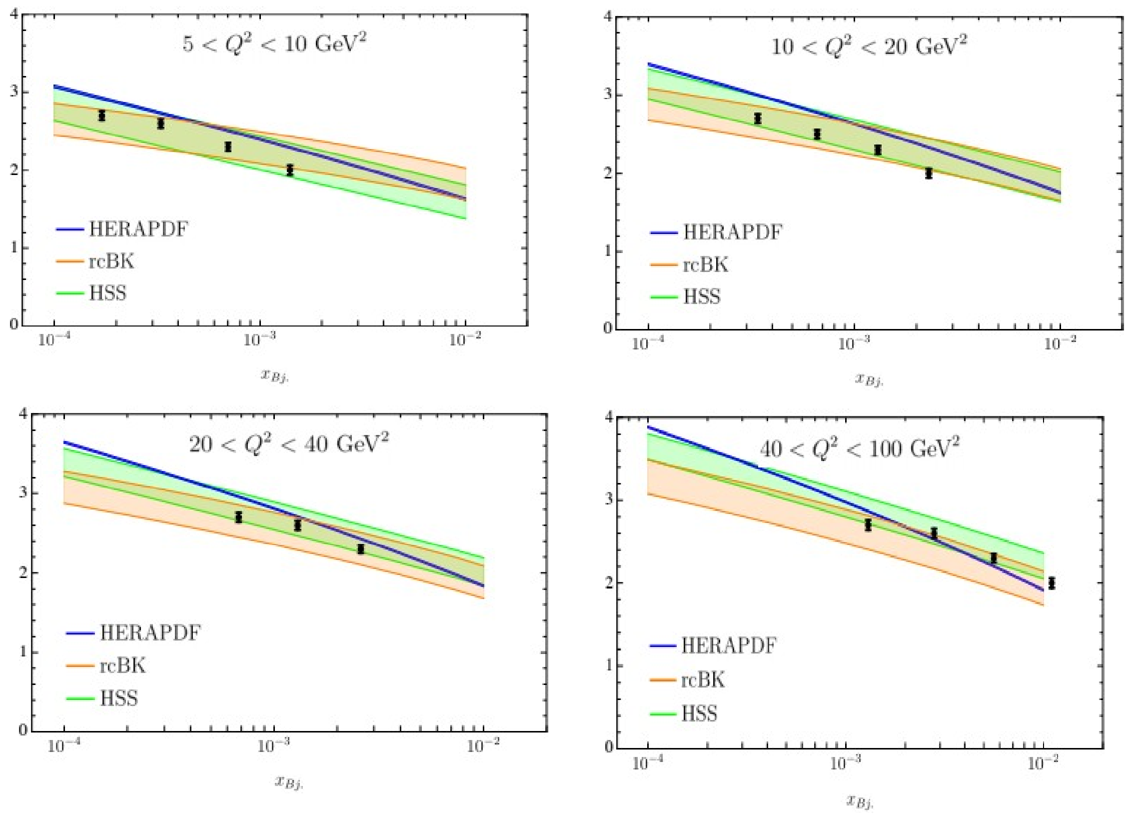


Hint that the general idea works. Gluon dominates over quarks.
One has to also take into account that only charged hadrons were measured i.e 2/3 of partons contribute

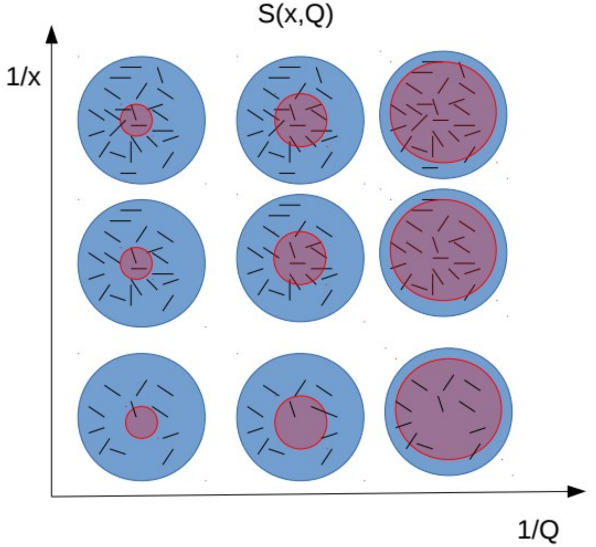
Large scales - description

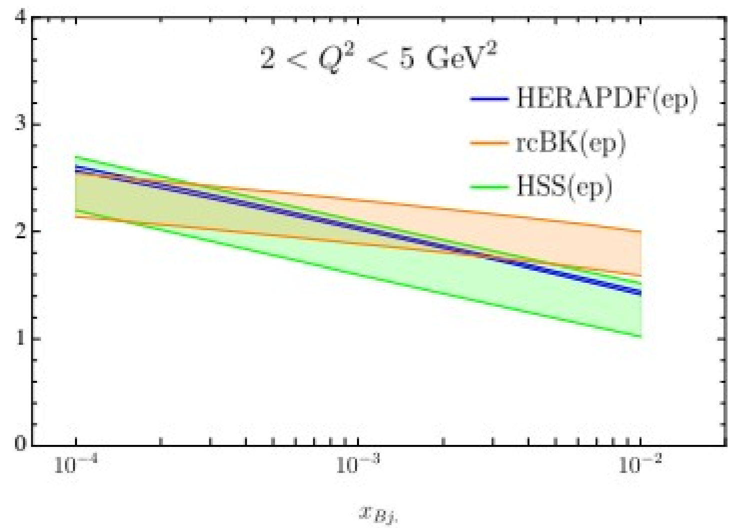
Martin Hentschinski, KK, Robert Straka '23

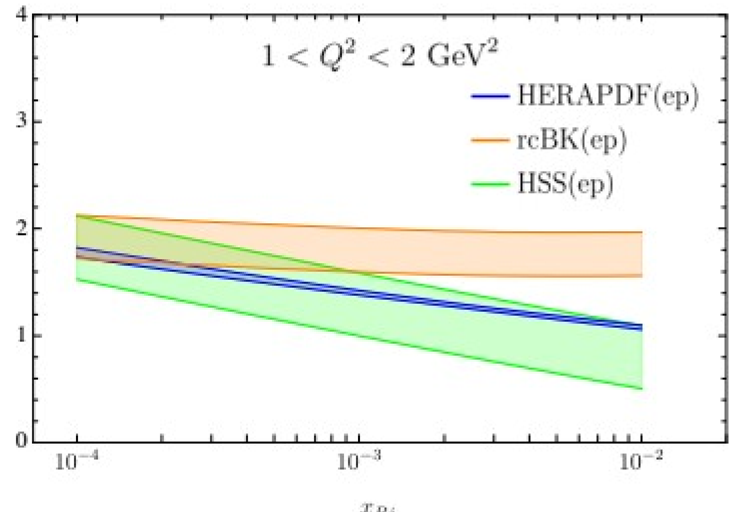




Small scales - prediction

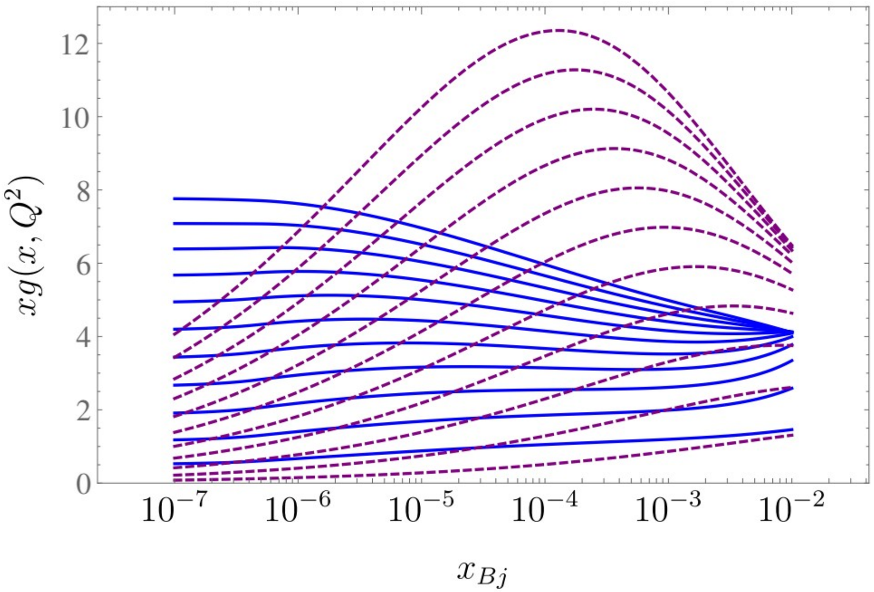


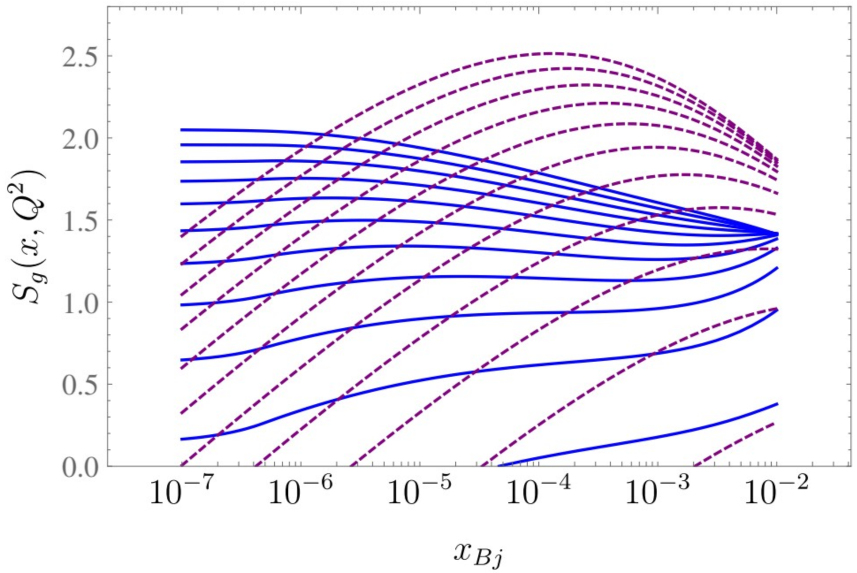




See also Hagivara, Hatta, Xiao ‘ 18
The genaralized KL model is used and entropy saturates in this approach
and Nowak, Liu, Zahed ‘22

Integrated gluon and entropy





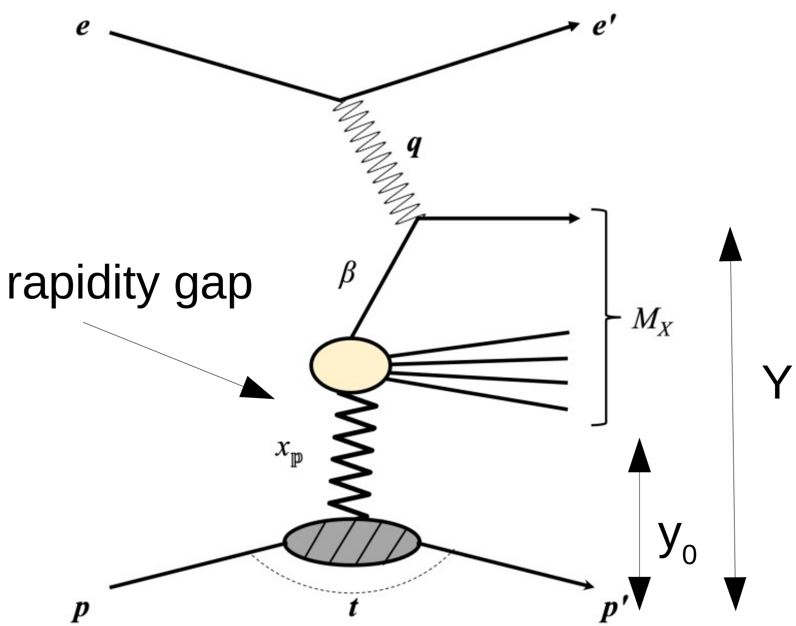
$$\lim_{Q^2 \gg Q_s^2} S(x, Q^2) = \ln\left(S_\perp Q_s^2(x)\right) + \ln\frac{N_c}{8\alpha_s \pi^2} = \lambda \ln\frac{1}{x} + \text{const}$$

$$\lim_{Q^2 \ll Q_s^2} S(x, Q^2) = \ln\left(\frac{S_\perp Q^4}{Q_s^2(x)}\right) + \ln\frac{N_c}{16\alpha_s \pi^2}$$

Photon can not resolve proton anymore therefore the EE vanishes.
But it might be that the formalism breaks down for low scales.
There might be another source of entropy that keep the total entropy not vanishing → generalized second law Bekenstein

H. Hentschinski, D. Kharzeev. K. Kutak, Z. Tu ‘23

EE in Diffractive Deep Inelastic Scattering



$x_{\mathbb{P}}$ proton’s momentum fraction carried by the Pomeron

β denotes the Pomeron’s momentum fraction carried by the quark interacting with the virtual photon

$x = \beta \cdot x_{\mathbb{P}}$ Bjorken x

$y_0 \simeq \ln 1/x_{\mathbb{P}}$ size of rpidity gap

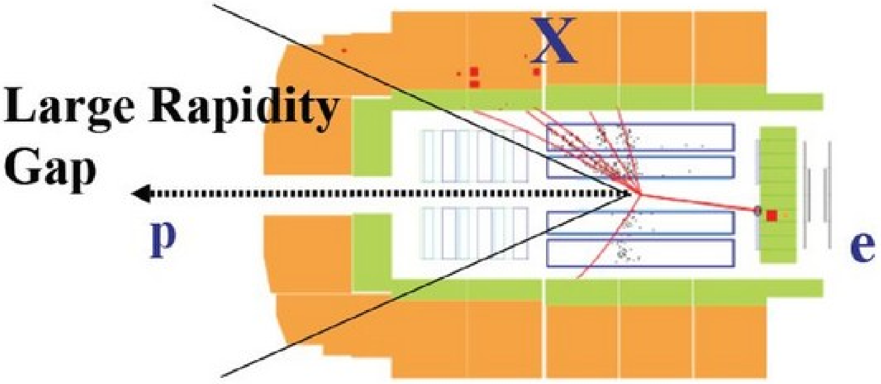
$Y = \ln 1/x$

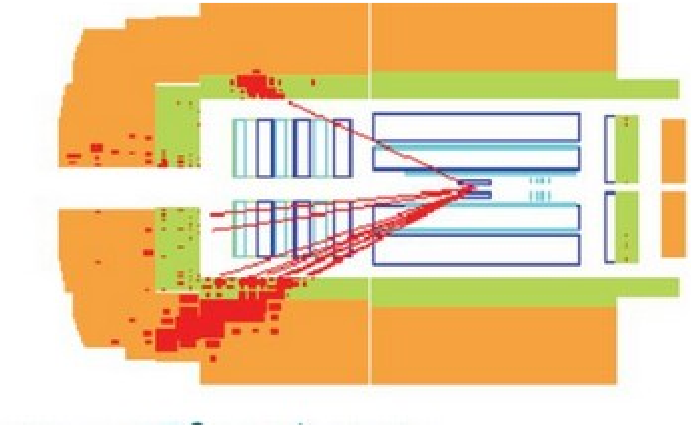
$y_X = Y - y_0 \simeq \ln 1/\beta$

Analogous evolution equation as for non-diffractive case but Initial conditions are different and there is delay because of rapidity gap.

Munier, Mueller Phys. Rev. D 98, 034021 (2018)

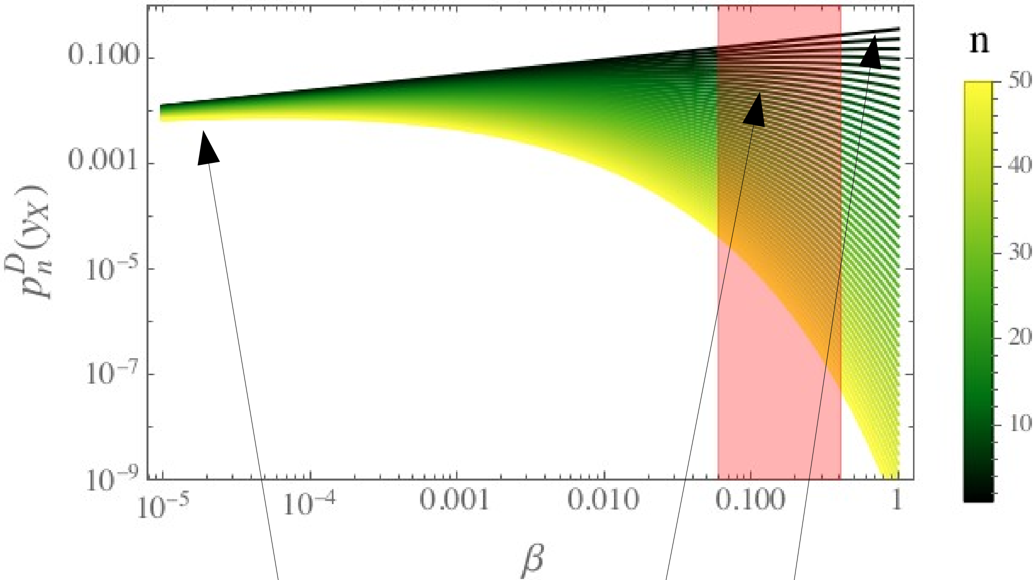
Diffraction vs. nondiffraction





H1 detector

EE in DDIS



$$\left\langle \frac{dn(\beta)}{d\beta} \right\rangle_{\text{charged}} \simeq \frac{2}{3} \left\langle \frac{dn(\beta)}{d\beta} \right\rangle$$

$$p_n^D(y_X) = \frac{1}{C} e^{-\Delta y_X} \left(1 - \frac{1}{C} e^{-\Delta y_X}\right)^{n-1}$$

$$p_n \equiv 1/Z$$

H1 data region

$$S(Z) = -\sum_n p_n \ln p_n = (1 - Z) \ln \frac{Z-1}{Z} + \ln Z$$

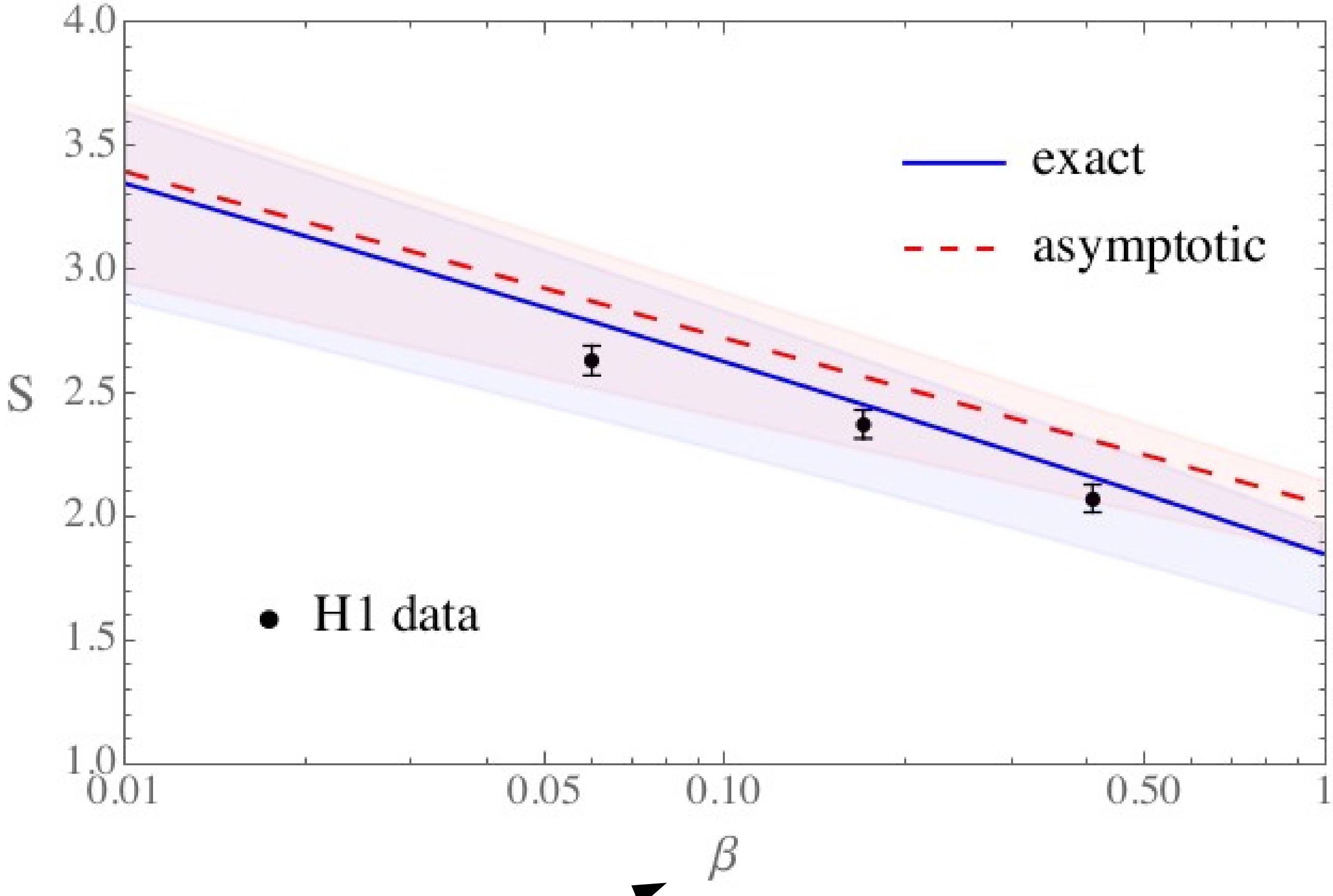
$$S_{\text{asym.}}(Z) = \ln Z + 1 + \mathcal{O}(1/Z)$$

Asymptotic or maximal entangled region. All configurations have the same probability

High probability of configurations with few partons

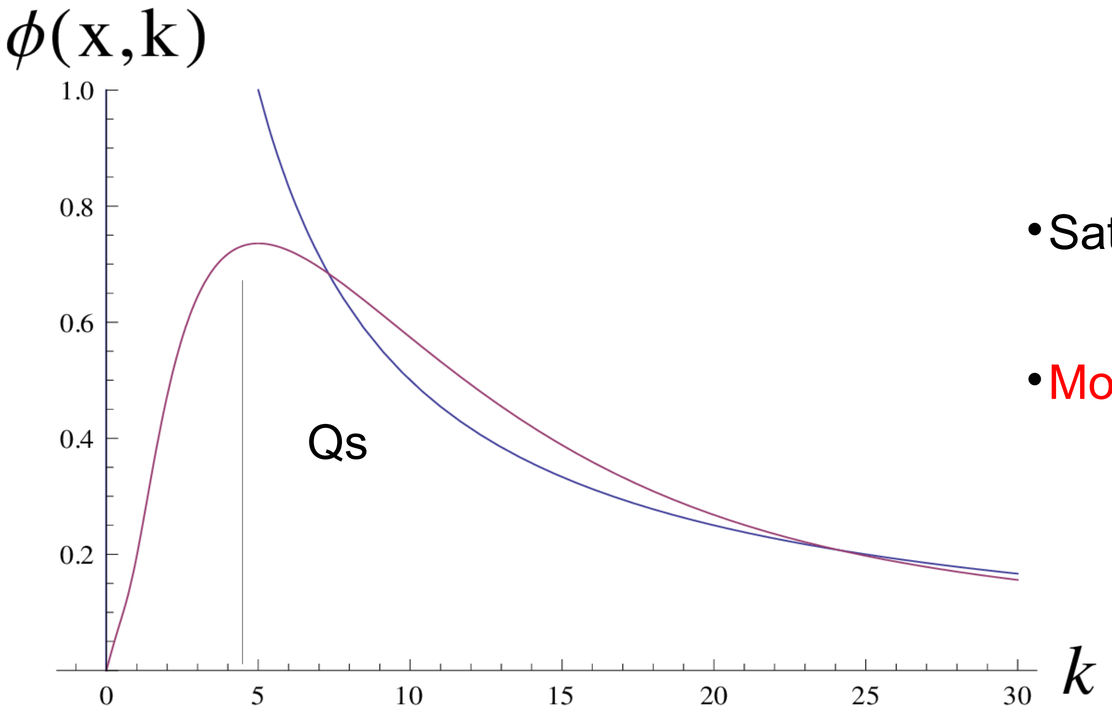
H. Hentschinski, D. Kharzeev. K. Kutak, Z. Tu ‘23

EE in DDIS



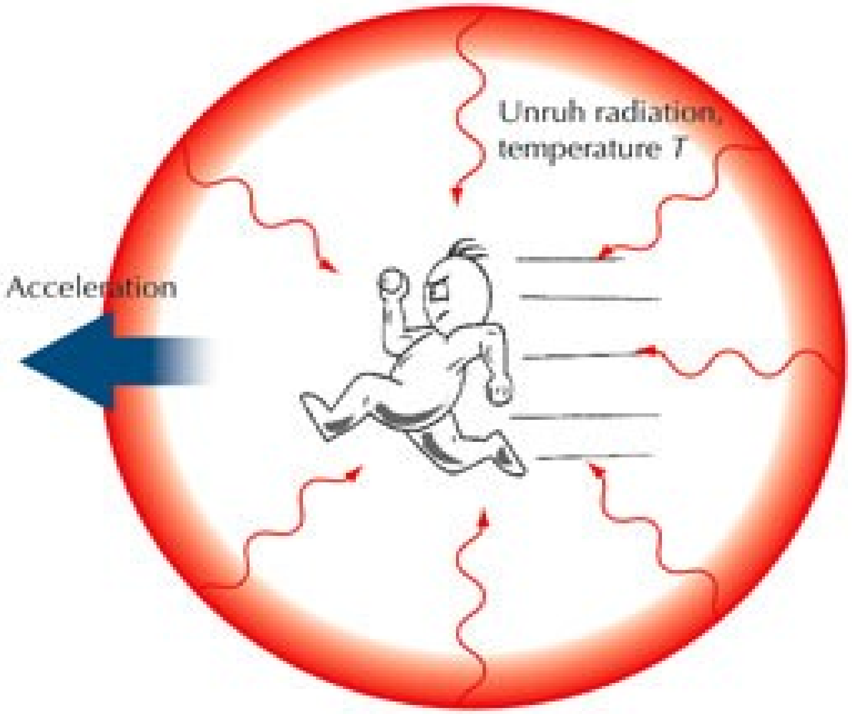
momentum fraction carried by the
quark interacting with the virtual photon

Saturation and gluon density



- Saturation scale regulates the divergence
- Most of gluons have momentum of the order of Qs

Unruh effect



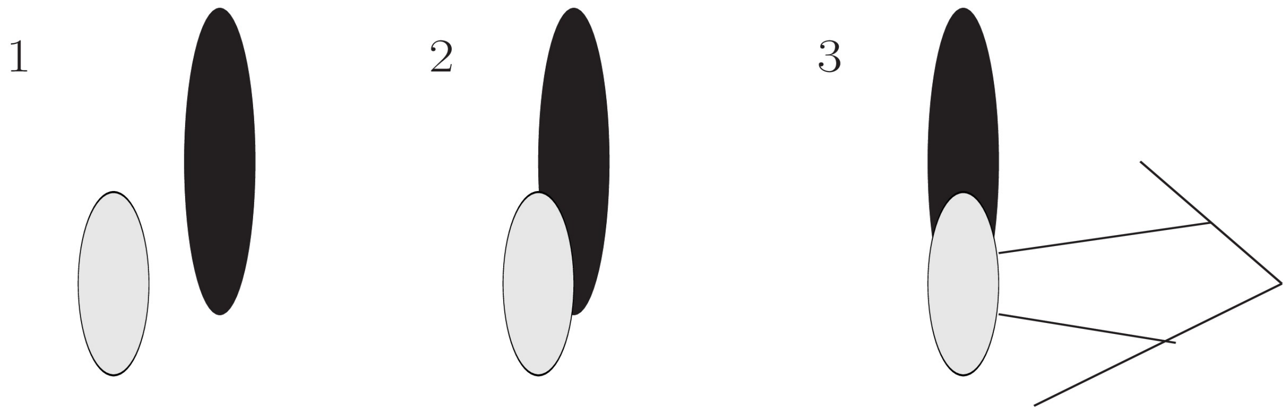
Accelerated observer in its rest frame feels thermal radiation or Bose-Einstein distribution with temperature

$$T = \frac{|a|}{2\pi}$$

Colliding hadrons and Unruh effect

Stages of collision

Kharzeev, Tuchin '05



Hadron of mass m after interaction with external field transforms into hadrons of mass M

$$P(M \leftarrow m) = 2\pi |\mathcal{T}(M \leftarrow m)|^2 \, \rho(M).$$

$$\int dM \, P(M \leftarrow m)$$

probability for transition to final state

should be finite

density of states determined by typical momentum. Qs emerges

$$|\mathcal{T}(M \leftarrow m)|^2 \sim \exp(-2\pi M/a)$$

transition amplitude

$$T = \frac{Q_s(x)}{2\pi}$$

Saturation and entropy

The relation $T = \frac{Q_s(x)}{2\pi}$ Can be understood in a generalized sense i.e. that saturation scale defines some temperature.

Equilibrium thermodynamics relations → Lower bound on produced entropy

It can be shown that the saturation line has an interpretation of a characteristics i.e. line along which the gluon density has a constant value.

Kutak 1103.3654.v1 and v2

$$dE = TdS$$

$$dE = dM$$

$$dM = TdS$$

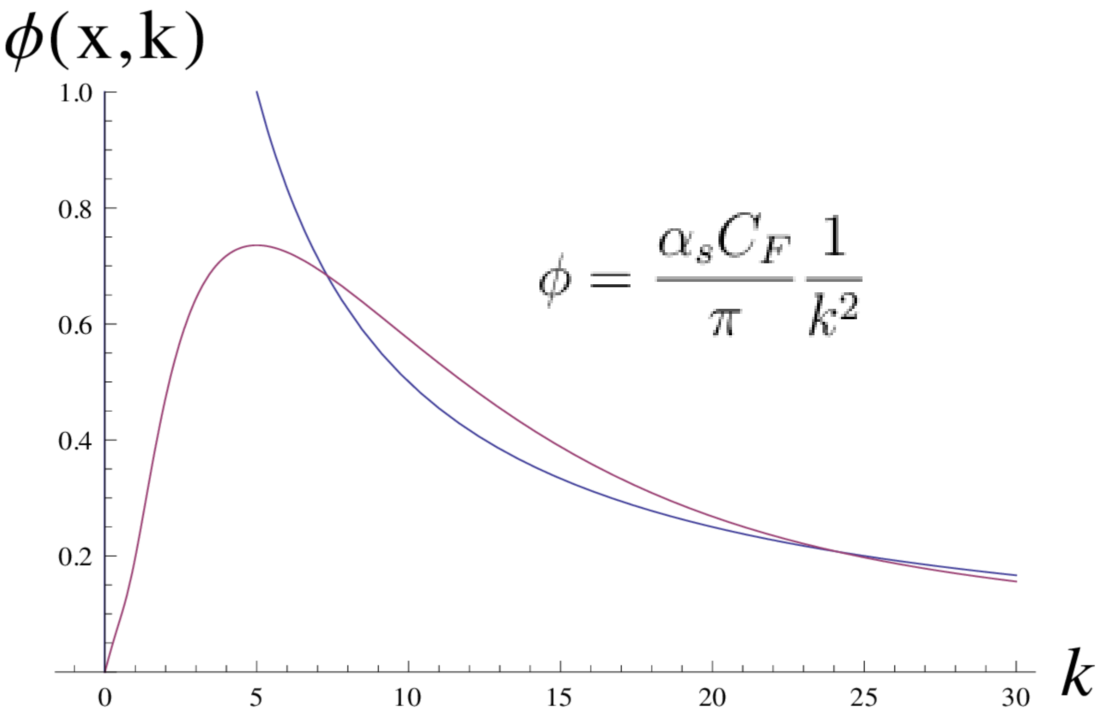
$$dM = dQ_s(x)$$

mass of system of gluons

$$\frac{dQ_s(x)}{Q_s(x)} = \frac{dS}{2\pi}$$

Gluon density and entropy

Kutak 1103.3654.v1 and v2



$$\Delta S = \pi \lambda \Delta y$$

$$\Delta y = \ln(x_0/x)$$

$$Q_s^2 = Q_0^2 (x_0/x)^\lambda$$

Many-body interactions

Medium generated mass of gluon.
Framework of Hard Thermal Loops.

Similarly in QED. Cut on photon`s kt
Is equivalent to introducing mass.

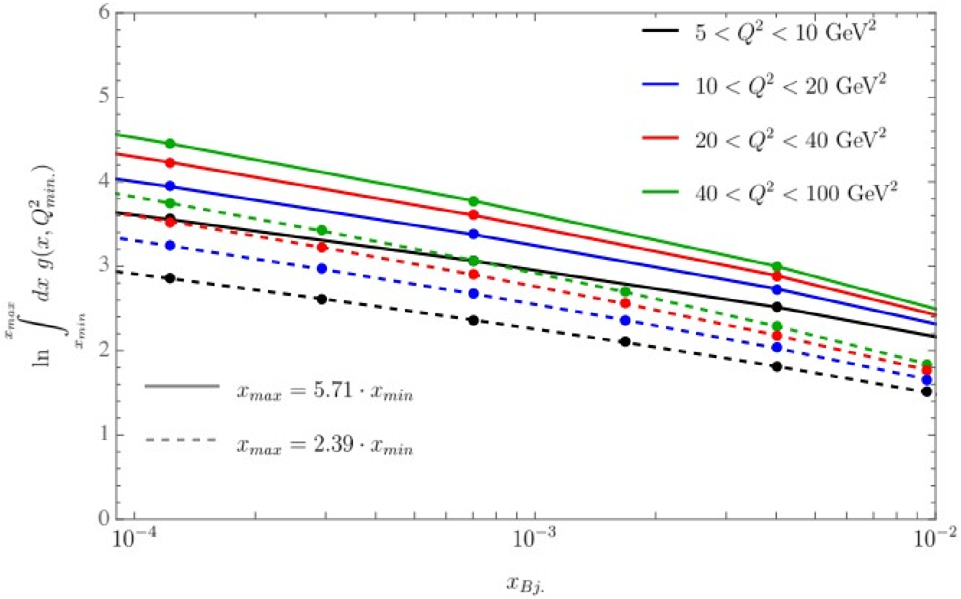
In presented approach mass is not fixed it is x dependent

Conclusions and outlook

- We show evidences for the proposal for low x maximal entanglement entropy of proton constituents .
- It can be systematically improved (quark contributions, NLO BFKL, rc BK) and can describe successfully H1 data.
- We obtain saturation of entropy at small resolution scales.
- We demonstrate that the proposal works for DDIS and that it can be used to study onset of maximal entanglement
- Saturation might be manifestation of prvide mechanism for vanishing of EE at low resolution scale
- The thermodynamic based approach agrees with KL approach

Backup

Bining and KL formula



plot showing dependence of the result on the sie of bins if binig is naive

$$n_g(Q^2) = \int_0^1 dx g(x, Q^2)$$

Formal definition of number of gluons

$$n_g(\bar{x}) = \int_{x_{\min}}^{x_{\max}} dx g(x, Q^2)$$

$\bar{x} \in [x_{\min}, x_{\max}]$

$$\bar{x} = \frac{\int_{x_{\min}}^{x_{\max}} dx\, x g(x, Q^2)}{\int_{x_{\min}}^{x_{\max}} dx g(x, Q^2)}$$

average *x*

Data binning takes place in rapidity

$$\bar{n}_g(\bar{x}) = \frac{1}{y_{max} - y_{min}} \int_{y_{min}}^{y_{max}} dy \frac{dn_g}{dy} = \frac{n_g(y_{max}) - n_g(y_{min})}{y_{max} - y_{min}}$$

$$y_{max,min} = \ln 1/x_{min,max}$$

for small bins

$$\bar{n}_g(x, Q^2) = \frac{dn_g}{d\ln(1/x)} = xg(x, Q^2)$$

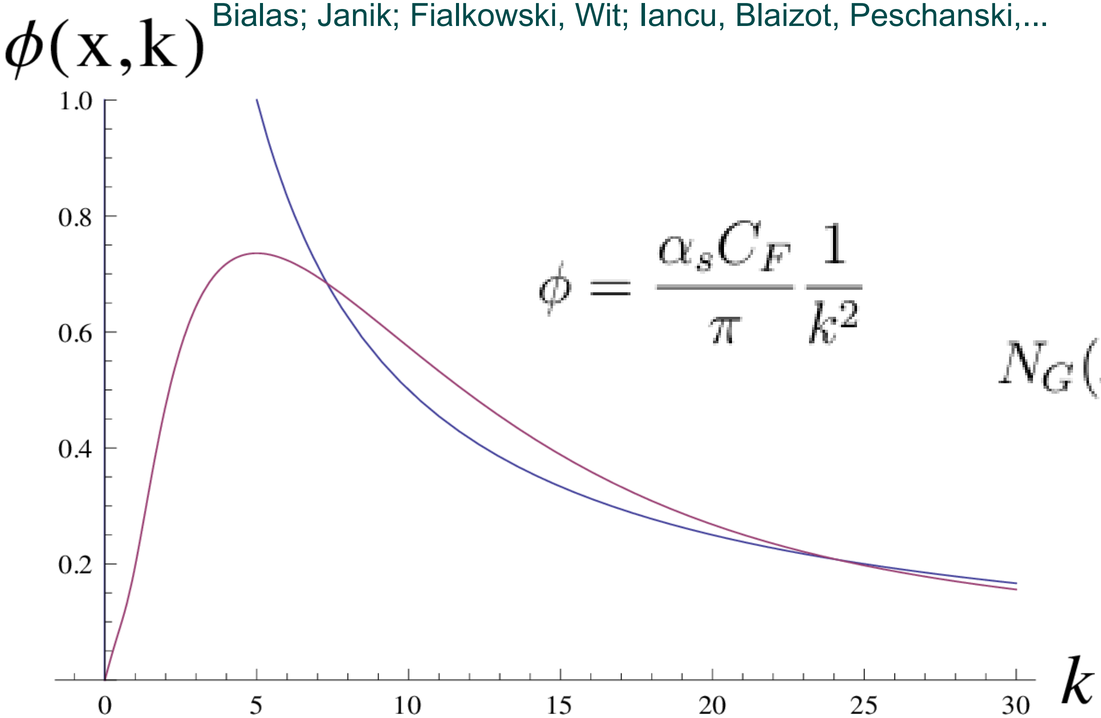
$$\langle \bar{n}(x, Q^2) \rangle_{Q^2} = \frac{1}{Q^2_{\max} - Q^2_{\min}} \int_{Q^2_{\min}}^{Q^2_{\max}} dQ^2 \left[xg(x, Q^2) + x\Sigma(x, Q^2) \right]$$

$$\langle S(x, Q^2) \rangle_{Q^2} = \ln \langle \bar{n}(x, Q^2) \rangle_{Q^2}$$

Gluon production and entropy – another assumptions

Kutak '11

Bialas; Janik; Fialkowski, Wit; Iancu, Blaizot, Peschanski,...



$M_G(x) = Q_s(x)$ energy dependent gluon's mass

$M(x) = N_G(x) M_G(x)$ mass of system of gluons

$N_G(x) \equiv \frac{dN}{dy} = \frac{1}{S_\perp}\frac{d\sigma}{dy}$ number of gluons

$$dE = TdS$$

$$dM = TdS$$

$$d\left[N_G(x)\, M_G(x)\right] = \frac{Q_s(x)}{2\pi} dS$$

Many-body interactions

Medium generated mass of gluon.
Framework of Hard Thermal Loops.

Entropy due to less dense hadron

$$S = \frac{6 C_F A_\perp}{\pi \alpha_s} Q_s^2(x) + S_0$$

$$S = 3\pi\left[N_G(x) + N_{G0}\right]$$

Similarly in QED. Cut on photon`s kt
Is equivalent to introducing mass.

In presented approach mass is not fixed it is x dependent