Equivalence between first-order causal and stable hydrodynamics and Israel-Stewart theory for boost-invariant systems with a constant relaxation time

Arpan Das

Institute of Nuclear Physics Polish Academy of Sciences Krakow, Poland

In collaboration with: W. Florkowski, J. Noronha, R. Ryblewski. Talk based on arXiv:2001.07983.

Outline

Introduction

- High energy heavy ion collisions [1] offer the opportunity to study the properties of hot and dense QCD matter.
- High energy heavy ion collisions is the only terrestrial laboratory where one can study a non abelian gauge theory in a nonperturbative regime.
- To understand the QCD matter we must know its space-time evolution.
- The space time evolution is affected by equation of state as well as dissipative, non equilibrium processes.
- Relativistic hydrodynamics has become nowadays the basic theoretical tool for modeling relativistic heavy-ion collisions [2].
- We need information of various transport coefficients such as viscosities, conductivities, and diffusivities.

[1] Florkowski, W.. Phenomenology of Ultra-Relativistic Heavy-IonCollisions. 2010.

[2] A. Muronga, PHYSICAL REVIEW C 69, 034903 (2004).

Relativistic Hydrodynamics

- Macroscopic evolution of the conserved quantities [3].

$$\partial_\mu T^{\mu\nu} = 0, \;\; \partial_\mu N^\mu = 0. \tag{1}$$

- An ideal fluid is defined by the assumption of local thermal equilibrium, i.e., all fluid elements must be exactly in thermodynamic equilibrium.

- Primary fluid-dynamical variables: $T(x)$, $\mu(x)$ and $u^\mu(x)$.

- The conserved currents of an ideal fluid can then be expressed as,

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu}, \;\; N^\mu = nu^\mu; \;\; S^\mu = su^\mu. \tag{2}$$

- $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projector orthogonal to u^μ.

[3] Romatschke, P., Romatschke, U.., arXiv:1712.05815.

- Corner stones of the formalism of ideal hydrodynamics are [4,5]
  1. Lorentz transformation.
  2. Conservation laws.
  3. Local thermodynamic equilibrium $\rightarrow$ a strong restriction.
- Dissipative effects in a fluid originate from irreversible thermodynamic processes that occur during the motion of the fluid.
- Exchange of heat with between fluid elements, relative motion between the fluid elements giving rise to dissipate energy by friction.
- All these processes must be included in order to obtain a reasonable description of a relativistic fluid.

[4] A. Jaiswal, arXiv:1408.0867

[5] A. Jaiswal, V. Roy, arXiv:1605.08694.

1st order Hydrodynamics

- The earliest covariant formulation of dissipative fluid dynamics were due to Eckart in 1940 [6] and, later, by Landau and Lifshitz in 1959 [7].
- Important to note
  1. For dissipative fluids, the energy-momentum tensor is no longer diagonal and isotropic in the local rest frame.
  2. Due to diffusion, the particle flow is expected to appear in the local rest frame of the fluid element.

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} + \Pi\Delta^{\mu\nu} + 2u^{(\mu} h^{\nu)} + \pi^{\mu\nu} \tag{3}$$

$$N^\mu = nu^\mu + n^\mu . \tag{4}$$

Π is the bulk viscous pressure, h^μ is the energy-diffusion current , $\pi^{\mu\nu}$ is the shear stress tensor.

[6] C. Eckart, Phys. Rev.58, 267 (1940).

[7] L.D. Landau and E.M. Lifshitz,Fluid Mechanics(Butterworth-Heinemann, Oxford,1987).

- In the theory of dissipative fluid dynamics, the important step is to fix u^μ.
  1. Eckart definition: velocity is defined by the flow of particles:
  $$N^\mu = nu^\mu, \quad n^\mu = 0. \tag{5}$$
  2. Landau definition: velocity is specified by the flow of the energy.
  $$u_\mu T^{\mu\nu} = \epsilon u^\nu \implies h^\mu = 0. \tag{6}$$
  3. In the Landau frame,
  $$T^{\mu\nu} = \epsilon u^{\mu\nu} + (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}, \tag{7}$$
  $$N^\mu = nu^\mu + n^\mu. \tag{8}$$
- In order to derive the complete set of equations for dissipative fluid dynamics along with the conservation equation we also need the dynamical or constitutive relations satisfied by the dissipative tensors, Π, $\pi^{\mu\nu}$ and n^μ.

2nd law of thermodynamics

- Entropy four current,

$$S^{\mu} = P\beta^{\mu} + \beta_{\nu} T^{\mu\nu} - \alpha N^{\mu}, \quad \beta^{\mu} = u^{\mu}/T \tag{9}$$

- The relativistic Navier-Stokes theory can be obtained by applying the second law of thermodynamics to each fluid element.

$$\partial_{\mu} S^{\mu} = -\beta \Pi \theta - n^{\mu} \nabla_{\mu} \alpha + \beta \pi^{\mu\nu} \sigma_{\mu\nu} \geq 0. \tag{10}$$

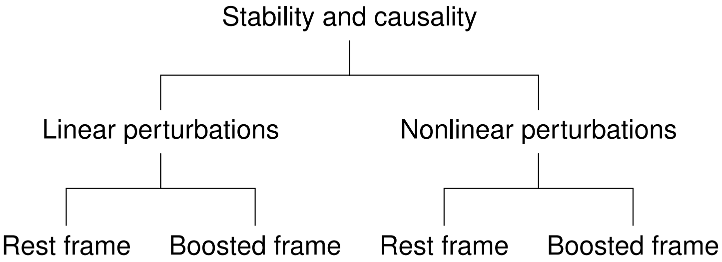
- Second law of thermodynamic can easily be satisfied if one identifies,

$$\Pi = -\zeta\theta; \quad n^{\mu} = \kappa \nabla^{\mu} \alpha, \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}. \tag{11}$$

- As long as $\zeta, \kappa, \eta \geq 0$, the entropy production is always positive.

Stability and causality

- Entropy production is essential but not sufficient condition for a theory of dissipative relativistic hydrodynamics[8,9].
- Dynamics of departures of these fluids from their equilibrium states or stability and causality also important for a relativistic theory.



- If the stability and causality is preserved in all the Lorentz boosted frames then we get an acceptable physical theories of relativistic dissipative hydrodynamics.

[8] W. A. Hiscock and L. Lindblom, ANNALS OF PHYSICS 151, 466-496 (1983)

[9] W. A. Hiscock and L. Lindblom,PHYSICAL REVIEW D, VOLUME 31, NUMBER 4, 725.

- Assume the background equilibrium state is homogeneous in space and the background is Minkowski space time.
- Look only for exponential plane wave solutions to the perturbation equations,

$$\delta Q = \delta Q_0 \exp(ikx + \Gamma t). \tag{12}$$

- The set of perturbation equation takes the form,

$$M^A_B \delta Y^B = 0 \tag{13}$$

  δY^B represents the list of fields which describe the perturbation of the fluid. M^A_B complex- valued matrix which describes the linearized equations of motion.
- There will exist exponential plane-wave solutions whenever Γ and k have values which satisfy the dispersion relation,

$$\det M = 0 \rightarrow \text{Block diagonal form} \rightarrow \text{shear mode and sound mode}. \tag{14}$$

IS theory

- A more realistic description of the entropy four-current can be obtained by considering it to be a function not only of the primary fluid-dynamical variables, but also of the dissipative currents.

$$S^\mu = P\beta^\mu + \beta_\nu T^{\mu\nu} - \alpha N^\mu - Q^\mu(\delta N^\mu, \delta T^{\mu\nu}) \tag{15}$$

- Up to second order in dissipative currents,

$$\begin{aligned} S^\mu = su^\mu - \alpha n^\mu - (\beta_0 \Pi^2 - \beta_1 n_\mu n^\mu + \beta_2 \pi_{\rho\sigma}\pi^{\rho\sigma})\frac{u^\mu}{2T} \\ -(\alpha_0 \Pi \Delta^{\mu\nu} + \alpha_1 \pi^{\mu\nu})\frac{n_\nu}{T} + \mathcal{O}(\delta^3). \end{aligned} \tag{16}$$

- In IS theory dissipative currents satisfy dynamical equations [10,11],

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\frac{1}{\beta_0}\Big[\theta + \beta_{\Pi\Pi}\Pi\theta + \alpha_0\nabla_\mu n^\mu + \psi\alpha_{n\Pi} n_\mu \dot{u}^\mu + \psi\alpha_{\Pi n} n_\mu \nabla^\mu \alpha\Big], \quad (17)$$

$$\dot{n}^{\langle\mu\rangle} + \frac{n^\mu}{\tau_n} = \frac{1}{\beta_1}\Big[T\nabla_\mu\alpha - \beta_{nn} n_\mu\theta + \alpha_0\nabla_\mu\Pi + \alpha_1\nabla_\nu\pi^\nu_\mu + \tilde{\psi}\alpha_{n\Pi}\Pi\dot{u}_\mu$$
$$+ \tilde{\psi}\alpha_{\Pi n}\Pi\nabla_\mu\alpha + \tilde{\chi}\alpha_{\pi n}\pi^\nu_\mu\nabla_\nu\alpha + \tilde{\chi}\alpha_{n\pi}\pi^\nu_\mu\dot{u}_\nu\Big] \quad (18)$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = \frac{1}{\beta_2}\Big[\sigma_{\mu\nu} - \beta_{\pi\pi}\theta\pi_{\mu\nu} - \alpha_1\nabla_{\langle\mu}n_{\nu\rangle} - \chi\alpha_{\pi n} n_{\langle\mu}\nabla_{\nu\rangle}\alpha$$
$$- \chi\alpha_{n\pi} n_{\langle\mu}\dot{u}_{\nu\rangle}\Big]. \quad (19)$$

- These relaxation times indicate the time scales within which the dissipative currents react to hydrodynamic gradients, in contrast to the relativistic Navier-Stokes theory.

[10] W. Israel and J. M. Stewart, “Transient relativistic thermodynamics and kinetic theory,” Annals Phys.118, 341 (1979).

[11] G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, Phys. Rev.D85(2012) 114047

- Important to note:
  1. In Landau's theory of relativistic viscous hydrodynamics deals with dynamical variables T, u^{α}, and μ ! but no causality and stability.
  2. IS theory deals with more variables Π, $\pi^{\mu\nu}$ and n^{μ}! but causality and stability is present.
- Is it possible to get a description of relativistic causal and stable viscous hydrodynamic description which only deals with Navier-Stokes degrees of freedom? $\rightarrow$ Yes.
- For conformal symmetric systems it is possible[12].
- For non conformal systems it is also possible[13] .

[12] Bemfica, F.S., Disconzi, M.M., Noronha, J.arXiv:1708.06255.

[13] Kovtun, P.arXiv:1907.08191

FOCS

- The physical objects $T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle$, $J^{\mu} = \langle \hat{J}^{\mu} \rangle$ can still be expressed in terms of the quantities T, u^{α} and μ.
- In equilibrium, the system can be parameterized by the temperature T, the timelike velocity vector u^{α}, and by the chemical potential μ.
- The equilibrium energy-momentum tensor and the current can be expressed in terms of T, u^{α}, and μ.
- However, out of equilibrium, T, u^{α} and μ have no first-principles microscopic definitions, and thus should be viewed as merely auxiliary variables used to parameterize the physical observable $T^{\mu\nu}$ and J^{μ}.

- The hydrodynamic expansion is a gradient expansion.

$$T^{\mu\nu} = \mathcal{O}(1) + \mathcal{O}(\partial) + \mathcal{O}(\partial^2) + ... + \mathcal{O}(\partial^k) + ..., \tag{20}$$

$$J^{\mu} = \mathcal{O}(1) + \mathcal{O}(\partial) + \mathcal{O}(\partial^2) + ... + \mathcal{O}(\partial^k) + ..., \tag{21}$$

where $\mathcal{O}(\partial^k)$ denotes the terms with k derivatives of T, u^α, μ, for example the $\mathcal{O}(\partial^2)$ contributions contain terms proportional to $\partial^2 T$, $(\partial T)^2$, $(\partial T)(\partial u)$ etc.

- Given a time like unit vector u^μ, the energy-momentum tensor ($T^{\mu\nu}$) and the current (J^μ) may be decomposed as,

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (Q^\mu u^\nu + Q^\nu u^\mu) + \mathcal{T}^{\mu\nu}, \quad J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu; \tag{22}$$

- Scalar basis: $u^\lambda\partial_\lambda T$, $u^\lambda\partial_\lambda(\mu/T)$, $\partial.\mu$,
- Vector basis: $\dot{u}^\mu, \Delta^{\mu\lambda}\partial_\lambda T, \Delta^{\mu\lambda}\partial_\lambda(\mu/T)$,
- Tensor basis: $\sigma^{\mu\nu}$.

- To the first order in the derivative expansion,

$$\mathcal{E} = \epsilon + \varepsilon_1 \frac{u^\lambda \partial_\lambda T}{T} + \varepsilon_2(\partial.u) + \varepsilon_3 u^\lambda \partial_\lambda(\mu/T) + \mathcal{O}(\partial^2), \tag{23}$$

$$\mathcal{P} = p + \pi_1 \frac{u^\lambda \partial_\lambda T}{T} + \pi_2(\partial.u) + \pi_3 u^\lambda \partial_\lambda(\mu/T) + \mathcal{O}(\partial^2), \tag{24}$$

$$Q^\mu = \theta_1 \dot{u}^\mu + \frac{\theta_2}{T}\Delta^{\mu\lambda}\partial_\lambda T + \theta_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + \mathcal{O}(\partial^2), \tag{25}$$

$$\mathcal{T}^{\mu\nu} = -\eta\sigma^{\mu\nu} + \mathcal{O}(\partial^2), \tag{26}$$

$$\mathcal{N} = n + \nu_1 \frac{\dot{T}}{T} + \nu_2(\partial.u) + \nu_3 u^\lambda \partial_\lambda(\mu/T) + \mathcal{O}(\partial^2), \tag{27}$$

$$\mathcal{J}^\mu = \gamma_1 \dot{u}^\mu + \frac{\gamma_2}{T}\Delta^{\mu\lambda}\partial_\lambda T + \gamma_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + \mathcal{O}(\partial^2), \tag{28}$$

- At zero-derivative order, the constitutive relations are determined by the three independent parameters ϵ, p, and n which in general all depend on T and μ.
- At one-derivative order, there are sixteen transport coefficients (seven for uncharged fluids) $\varepsilon_{1,2,3}$, $\pi_{1,2,3}$, $\theta_{1,2,3}$, $\nu_{1,2,3}$, $\gamma_{1,2,3}$, and η.

- Given any choice of T, u^α and μ, one can always redefine,

$$T' = T + \delta T; \mu' = \mu + \delta\mu; u'^\alpha = u^\alpha + \delta u^\alpha. \tag{29}$$

$$\delta T = a_1 \dot{T}/T + a_2\partial\cdot u + a_3 u^\lambda \partial_\lambda(\mu/T)\,, \tag{30}$$

$$\delta u^\mu = b_1 \dot{u}^\mu + b_2/T\,\Delta^{\mu\nu}\partial_\nu T + b_3\Delta^{\mu\lambda}\partial_\lambda(\mu/T)\,, \tag{31}$$

$$\delta\mu = c_1 \dot{T}/T + c_2\partial\cdot u + c_3 u^\lambda \partial_\lambda(\mu/T)\,, \tag{32}$$

- Only criteria is that all these choices agree in equilibrium.
- In terms of the redefinition of the fundamental hydrodynamic variables, $\mathcal{E}$, $\mathcal{P}$, Q^μ, $\mathcal{N}$, $\mathcal{T}^{\mu\nu}$ and $\mathcal{J}^\mu$ all changes, but what remains unchanged are $T^{\mu\nu}$ and J^μ.

- The constitutive relations for $T^{\mu\nu}$ and J^{μ}, written in terms of the new fields T', u', μ', look the same as the constitutive relations in terms of the old fields T, u, μ, with the following change:

$$\varepsilon_i \to \varepsilon_i - \epsilon_{,T} a_i - \epsilon_{,\mu} c_i \,, \tag{33}$$

$$\pi_i \to \pi_i - p_{,T} a_i - p_{,\mu} c_i \,, \tag{34}$$

$$\nu_i \to \nu_i - n_{,T} a_i - n_{,\mu} c_i \,, \tag{35}$$

$$\theta_i \to \theta_i - (\epsilon + p) b_i \,, \tag{36}$$

$$\gamma_i \to \gamma_i - n b_i \,, \tag{37}$$

$$\eta \to \eta \,, \tag{38}$$

- It is clear that ε_i, π_i, ν_i, θ_i, γ_i are not invariant under redefinition of the fundamental variables. But one can construct some invariant quantities, e.g.

$$f_i = \pi_i - \left(\frac{\partial p}{\partial \epsilon}\right)_n \varepsilon_i - \left(\frac{\partial p}{\partial n}\right)_\epsilon \nu_i. \tag{39}$$

$$l_i \equiv \gamma_i - \frac{n}{\epsilon + p}\theta_i. \tag{40}$$

- Only invariant transport quantities are, f_i, l_i and η.

Getting back Landau frame

- One can get back Landau frame by setting, $\mathcal{E} = \epsilon$, $\mathcal{N} = n$, $Q^\mu = 0$.
- We can always make this choice by proper choice of T, μ and u^μ, i.e. a_i, b_i and c_i.
- For the choice of a_i, b_i and c_i which are consistent with $\mathcal{E} = \epsilon$, $\mathcal{N} = n$, $Q^\mu = 0$, it can be shown that,$\pi_i = f_i$ and $\gamma_i = l_i \rightarrow$ invariant quantities.
- In the Landau frame,

$$T^{\mu\nu} = \epsilon u^{\mu\nu} + \left(p + f_1\frac{\dot{T}}{T} + f_2\partial.u + f_3 u^\lambda\partial_\lambda(\mu/T)\right)\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} + \mathcal{O}(\partial^2), \tag{41}$$

$$J^\mu = nu^\mu + l_1\dot{u}^\mu + \frac{l_2}{T}\Delta^{\mu\lambda}\partial_\lambda T + l_3\Delta^{\mu\lambda}\partial_\lambda(\mu/T) + \mathcal{O}(\partial^2). \tag{42}$$

- Till now we have not used onshell conditions, i.e. $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu N^\mu = 0$.

- The conservation equations,

$$\partial_\mu(nu^\mu) + O(\partial^2) = 0, \quad u_\nu\partial_\mu(\epsilon u^\mu u^\nu + p\Delta^{\mu\nu}) + O(\partial^2) = 0 \tag{43}$$

  imply two "on-shell" relations among the scalars $\dot{T}$, $\partial{\cdot}u$, and $\dot{\mu}$, up to $O(\partial^2)$ terms.

- Similarly,

$$\Delta^\alpha_\nu\partial_\mu(\epsilon u^\mu u^\nu + p\Delta^{\mu\nu}) + O(\partial^2) = 0 \tag{44}$$

  implies one "on-shell" relation among the vectors $\dot{u}^\alpha$, $\Delta^{\alpha\lambda}\partial_\lambda T$, and $\Delta^{\alpha\lambda}\partial_\lambda(\mu/T)$, up to $O(\partial^2)$ terms.

- Using onshell relations,

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + [p - \zeta(\partial.u)]\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} + \mathcal{O}(\partial^2), \tag{45}$$

$$J^\mu = nu^\mu - \sigma T\Delta^{\mu\lambda}\partial_\lambda(\mu/T) + \chi_T\Delta^{\mu\lambda}\partial_\lambda T + \mathcal{O}(\partial^2), \tag{46}$$

  with,

$$\sigma = \frac{n}{\epsilon + p}l_1 - \frac{l_3}{T}; \quad \chi_T = \frac{1}{T}(l_2 - l_1). \tag{47}$$

- Important ingredients:
  1. The choice of frame such that $\mathcal{E} = \epsilon$, $\mathcal{N} = n$, $\mathcal{Q}^{\mu} = 0$.
  2. The on-shell relations derived from zero-derivative hydrodynamics.
- The arbitrariness involved in using the on sell condition implies that there are multiple ways to write down the on-shell constitutive relations.
- Landau frame is a whole class of frames.
- Eliminating different one-derivative quantities will lead to different Landau-frame hydrodynamic equations which may have different stability and causality properties.

- Study the linearized stability, $T = T_0 + \delta T$, $v = v_0 + \delta v$, $\mu = \mu_0 + \delta\mu$.
- Look for plane wave solutions of the form, $e^{ik.x - i\omega t}$.
- First-order hydrodynamics of uncharged fluids in the general frame we have six transport coefficients: $\varepsilon_{1,2}$, $\pi_{1,2}$, $\theta \equiv \theta_1 = \theta_2$, and η.
- For charged fluids in the general frame one has a fourteen-dimensional parameter space of transport coefficients, $\varepsilon_{1,2,3}$, $\pi_{1,2,3}$, $\nu_{1,2,3}$, $\theta \equiv \theta_1 = \theta_2$, θ_3, $\gamma \equiv \gamma_1 = \gamma_2$, γ_3, η.
- One can also find a subspace in the fourteen-dimensional parameter space of transport coefficients where a class of stable frames can be defined.

- Shear channel for uncharged fluid e.g.

$$\omega(k) = \frac{i(\epsilon_0 + p_0)\sqrt{1 - v_0^2}}{\eta v_0^2 - \theta} + O(k.v_0)\,, \tag{48}$$

- Stability of the shear channel fluctuations requires: $\theta > \eta > 0$.
- The Landau-Lifshitz convention sets $\theta = 0$ at non-zero $\eta \implies$ stability criteria is not satisfied.
- Sound channel of uncharged fluid e.g.

$$\omega(k) = -i\frac{\epsilon_0 + p_0}{\theta} + O(k^2). \tag{49}$$

- For the stability of the sound mode one requires $\theta > 0$.
- $\theta > 0$ contradicts the Landau-Lifshitz convention.
- Special frame choice of the most general first order hydrodynamics can give rise to unstable equilibrium state. This is just a bad choice of the frame. In general first order hydrodynamics is stable and causal.

IS-FOCS correspondence

- We have two fundamentally different frame work of causal and stable relativistic viscous hydrodynamics,
  1. First order hydrodynamics in general frame (FOCS).
  2. Second order Israel Stewart theory (IS).
- In IS theory stability and causality: relaxation type evolution equation of the dissipative fluxes Π, $\pi^{\mu\nu}$ and n^{μ}.
- In FOCS stability and causality: proper choice of hydrodynamic frame with more transport coefficients.
- Stability and causality in FOCS can be explored even in the non linear regime.
- So any mapping between IS and FOCS will allow us to explore IS theories with our knowledge of FOCS theory.

- For boost-invariant conformal systems (exact conformal symmetry) two approaches lead to very similar equations [14].
- We assume that the system's equation of state is conformal (massless particles) but we allow for a non-conformal behavior of the coefficients in the regulating sector of the theory.
- We show that if the kinetic coefficients are expressed in terms of a constant relaxation time there is an exact match between the dynamics described by FOCS and IS formulations.
- This allows us to get the first general analytical solution of the FOCS equations.

[14] Bemfica, F.S., Disconzi, M.M., Noronha, J.arXiv:1708.06255.

- In the IS sector we consider following equations [15,16]

$$D\epsilon + (\epsilon + p)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0, \tag{50}$$

$$(\epsilon + p)Du^{\mu} - \Delta^{\mu}_{\lambda}\nabla^{\lambda}P + \Delta^{\mu}_{\lambda}\nabla_{\mu}\pi^{\mu\lambda} = 0, \tag{51}$$

$$\tau_R\Delta^{\mu\nu}_{\alpha\beta}D\pi^{\alpha\beta} + \delta_{\pi\pi}\theta\pi^{\mu\nu} + \tau_{\pi\pi}\Delta^{\mu\nu}_{\alpha\beta}\pi^{\alpha\lambda}\sigma^{\beta}_{\lambda} - 2\tau_R\Delta^{\mu\nu}_{\alpha\beta}\pi^{\alpha}_{\lambda}\omega^{\beta\lambda} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}. \tag{52}$$

- For Bjorken flow, $ds^2 = d\tau^2 - dx^2 - dy^2 - \tau^2 d\xi^2$, $\tau = \sqrt{t^2 - \tau^2}$, $\xi = Tanh^{-1}(z/t)$, $u^{\mu} = (1,0,0,0)$, $\pi^{\mu}_{\nu} = diag(0, -\pi/2, -\pi/2, \pi)$, $\delta_{\pi\pi} = 4/3\tau_R, \tau_{\pi\pi} = \lambda\tau_R$.

- For the IS theory hydrodynamic equation becomes,

$$\frac{d\varepsilon}{d\tau} = -\frac{\varepsilon + p}{\tau} + \frac{\pi}{\tau}, \tag{53}$$

$$\tau_R\frac{d\pi}{d\tau} + \pi = \frac{4}{3}\frac{\eta}{\tau} - \left(\frac{4}{3} + \lambda\right)\tau_R\frac{\pi}{\tau}, \tag{54}$$

[15] Bjorken, J.D..Phys Rev1983,D27,140-151.

[16] G.S.Denicol and J. Noronha, arXiv:1711.01657

- Assuming the conformal equation of state $p = \frac{1}{3}\varepsilon = \frac{aT^4}{3}$, and introducing the variable

$$y = \frac{dT}{d\tau}, \tag{55}$$

allows us to rewrite Eq. (54) as,

$$4a\tau_R T^3 \frac{dy}{d\tau} + 12\tau_R a T^2 y^2 + aT^3 y\left[4 + \left(\frac{28}{3} + 4\left(\frac{4}{3} + \lambda\right)\right)\frac{\tau_R}{\tau}\right] + \frac{4aT^4}{3\tau} + \frac{4}{3}aT^4\left(\frac{4}{3} + \lambda\right)\frac{\tau_R}{\tau^2} - \frac{4}{3}\frac{\eta}{\tau^2} = 0. \tag{56}$$

- Equations (55) and (56) are coupled differential equations for the functions T and y, which are completely equivalent to the original IS equations.
- Note that Eq. (56) has the form of a Ricatti equation ($ay' + by^2 + cy + d = 0$, with $b/a \neq 0$ and $c/a \neq 0$), which was analyzed and may be possible to solve analytically. [17]

[17] G.S.Denicol and J. Noronha, arXiv:1711.01657

- For the FOCS approach, the evolution equations are reduced to the formula

$$\frac{d\mathcal{E}}{d\tau}+\frac{\mathcal{E}+\mathcal{P}}{\tau}-\frac{4}{3}\frac{\eta}{\tau^2}=0, \tag{57}$$

where the following constitutive relations are assumed,

$$\mathcal{E}=\epsilon+\varepsilon_1\frac{dT}{Td\tau}+\frac{\varepsilon_2}{\tau};\mathcal{P}=p+\pi_1\frac{dT}{Td\tau}+\frac{\pi_2}{\tau}. \tag{58}$$

- In natural units, the coefficient functions ε_1, ε_2, π_1, and π_2 have dimension of energy cubed, so for conformal systems they should scale as T^3.
- Similarly, in this case the IS relaxation time τ_R should be inversely proportional to T.
- For strictly conformal systems with Weyl transformations we require also that $\mathcal{E}=3\mathcal{P} \implies \pi_i=(1/3)\varepsilon_i$ along with $\varepsilon_1=3\varepsilon_2$.
- In this work we also discuss yet another case, where the coefficients ε_1, ε_2, π_1, and π_2 are expressed in terms of a constant relaxation time $\implies x_i=x_i^0T^4$.

- To include and discuss different cases together we take the constitutive relation in the following form,

$$\mathcal{E} = aT^4 + \varepsilon_1^0\, T^n \frac{dT}{Td\tau} + \frac{\varepsilon_2^0}{\tau}\, T^n ; \mathcal{P} = \frac{aT^4}{3} + \pi_1^0\, T^n \frac{dT}{Td\tau} + \frac{\pi_2^0}{\tau}\, T^n, \tag{59}$$

where ε_1^0, ε_2^0, π_1^0, and π_2^0 are dimensionless ($n = 3$) or dimensionful ($n \neq 3$) constants (for constant relaxation times).

- FOCS hydrodynamic equation with $y = dT/d\tau$,

$$\varepsilon_1^0 T^{n-1} \frac{dy}{d\tau} + (n-1)\,\varepsilon_1^0\, T^{n-2}\, y^2 + \left(4aT^3 + (\varepsilon_1^0 + \pi_1^0 + n\,\varepsilon_2^0)\frac{T^{n-1}}{\tau}\right) y + \frac{4}{3\tau} aT^4 + \frac{\pi_2\, T^n}{\tau^2} - \frac{4}{3}\frac{\eta}{\tau^2} = 0. \tag{60}$$

- we can formulate the IS and FOCS frameworks in terms of the two differential equations for the temperature T and its derivative $y = dT/d\tau$.

- Both IS and FOCS formalism has one common equation, $y = dT/d\tau$.
- After equating the terms with the same derivatives of the function y in Eq. (56) and (60) we find:

$$\begin{aligned}
\varepsilon_1^0 &= 4a\tau_R T^{4-n}, && (61)\\
\varepsilon_1^0 &= \frac{12}{n-1} a\tau_R T^{4-n}, && (62)\\
\pi_1^0 &= \frac{4}{3} a\tau_R (11+3\lambda) T^{4-n} - \varepsilon_1^0 - n\varepsilon_2^0, && (63)\\
\pi_2^0 &= \frac{4}{9} a\tau_R \left(4+3\lambda\right) T^{4-n}. && (64)
\end{aligned}$$

- One can easily notice that in the strictly conformal case, $n = 3$, it is impossible to exactly match the FOCS and IS equations.
- An interesting situation takes place when $n = 4$. In this case Eqs. (61) and (62) are fully consistent.
- The kinetic coefficient ε_1^0 has dimension of fm and, thus, it can be treated as a fixed relaxation time related to τ_R (which is also constant).

- To uniquely determine the kinetic coefficient in the FOCS theory we use the traceless condition of the energy momentum tensor.

$$T^{\mu}_{\mu} = 0 \implies \pi_i = \epsilon_i/3 \implies \lambda = -1. \tag{65}$$

- In the FOCS approach, the bulk viscosity appears as a linear combination of the regulators and one can show that [18],

$$\zeta = (-\pi_2 + c_s^2(\varepsilon_2 + \pi_1) - c_s^4 \varepsilon_1) = 0. \tag{66}$$

- For conformal equation of state with $c_s^2 = 1/3$ and the condition of vanishing of $T^{\mu\nu}$,

$$\varepsilon_1^0 = 4a\tau_R, \tag{67}$$

$$\varepsilon_2^0 = \frac{4}{3}a\tau_R, \tag{68}$$

$$\pi_1^0 = \frac{4}{3}a\tau_R, \tag{69}$$

$$\pi_2^0 = \frac{4}{9}a\tau_R. \tag{70}$$

[18] Kovtun, P,arXiv:1907.08191.

Analytical solution of FOCS

- The general solution of the IS equations in (53) and (54) in Bjorken flow is [19],

$$\varepsilon(\hat{\tau}) = \varepsilon_0 \left(\frac{\hat{\tau}_0}{\hat{\tau}}\right)^{\frac{4}{3}+\frac{\lambda+1}{2}} \exp\left(-\frac{\hat{\tau}-\hat{\tau}_0}{2}\right) \times \left[\frac{M_{-\frac{\lambda+1}{2},\frac{\sqrt{\lambda^2+4\tilde{a}}}{2}}(\hat{\tau}) + \alpha\, W_{-\frac{\lambda+1}{2},\frac{\sqrt{\lambda^2+4\tilde{a}}}{2}}(\hat{\tau})}{M_{-\frac{\lambda+1}{2},\frac{\sqrt{\lambda^2+4\tilde{a}}}{2}}(\hat{\tau}_0) + \alpha\, W_{-\frac{\lambda+1}{2},\frac{\sqrt{\lambda^2+4\tilde{a}}}{2}}(\hat{\tau}_0)}\right] \tag{71}$$

  where $\hat{\tau} = \tau/\tau_R$, $\tilde{a} = 16/(9\tau_R T)(\eta/s)$,$\hat{\tau}_0$ is the initial time, ε_0 and α are constants that define the initial value problem, and $M_{k,\mu}(x)$ and $W_{k,\mu}(x)$ are Whittaker functions.
- The matching to IS theory shown here implies that the general solution for the energy density in IS also holds for the FOCS theory (for appropriate values of λ).
- This is the first analytical solution of the FOCS theory.

[19] Denicol, G.S., Noronha, J.. arXiv:1711.01657.

- For boost-invariant, baryon-free systems with a conformal equation of state, If the regulator sectors of the theories are determined by a constant relaxation time, there exists a mapping between the FOCS and IS approaches that makes their dynamics exactly the same.
- The causality conditions for the FOCS approach found here proved to be relevant when determining the range of acceptable values of the transport coefficients in the FOCS approach, after the matching to IS theory.
- Mapping between IS theory and FOCS theory is only well defined if the IS parameter λ takes values that are distinct from the standard 14-moment result.
- For a more complex system simple correspondence between FOCS and IS may not exists.
- FOCS yields four second-order equations which are in general equivalent to eight first-order equations, while IS is based on ten equations describing the time evolution of ten independent components of the symmetric energy-momentum tensor.

Thank You!

Perturbation Hydro equations

$$
\begin{aligned}
T^{ab} &= \rho u^a u^b + (p+\tau) q^{ab} + q^a u^b + q^b u^a + \tau^{ab}, && (72)\\
N^a &= n u^a + \nu^a, && (73)\\
\nabla_a \delta T^{ab} &= 0, && (74)\\
\nabla_a \delta N^a &= 0, && (75)\\
\delta\tau &= -\zeta \nabla_a \delta u^a && (76)\\
\delta q^a &= -\kappa T q^{ab} \left(\nabla_b(\delta T/T) + u^c \nabla_c \delta u_b + \delta u^c \nabla_c u_b\right), && (77)\\
\delta\nu^a &= -\sigma T^2 q^{ab} \nabla_b \delta\Theta, && (78)\\
\delta\tau^{ab} &= -2\eta \langle \nabla^a \delta u^b + \delta u^a u^c \nabla_c u^b \rangle, && (79)\\
\delta T^{ab} &= (\rho+p)(\delta^a u^b + u^a \delta u^b) + \delta\rho u^a u^b + (\delta\rho + \delta\tau) q^{ab}\\
&\quad + u^a \delta q^b + u^b \delta q^a + \delta\tau^{ab},\\
\delta N^a &= \delta n u^a + n \delta u^a + \delta\nu^a. && (80)
\end{aligned}
$$

- To discuss the notion of stability and causality at the non linear level,

$$T^{\mu\nu} = (\varepsilon + \mathcal{A}_1)u^\mu u^\nu + (P(\varepsilon) + \mathcal{A}_2)\,\Delta^{\mu\nu} - 2\eta\sigma^{\mu\nu} + u^\mu \mathcal{Q}^\nu + u^\nu \mathcal{Q}^\mu, \quad (81)$$

$$\mathcal{A}_1 = \chi_1 \frac{u^\alpha \nabla_\alpha \varepsilon}{\varepsilon + P} + \chi_2 \nabla_\alpha u^\alpha, \mathcal{A}_2 = \chi_3 \frac{u^\alpha \nabla_\alpha \varepsilon}{\varepsilon + P} + \chi_4 \nabla_\alpha u^\alpha,$$
$$\mathcal{Q}_\mu = \lambda_{BDN} \left(\frac{c_s^2 \Delta_\mu^\nu \nabla_\nu \varepsilon}{\varepsilon + P} + u^\alpha \nabla_\alpha u_\mu \right). \quad (82)$$

- The coefficients χ_i can be directly related to our parametrizations ($n = 4$):

$$3\chi_1 = \varepsilon_1^0 T^4,\ \chi_2 = \varepsilon_2^0 T^4,\ 3\chi_3 = \pi_1^0 T^4,\ \chi_4 = \pi_2^0 T^4. \quad (83)$$

- For the Bjorken case $\mathcal{Q}_\mu$ does not appear in the FOCS equation. Thus in this case λ_{BDN} remains unconstrained.

- Causality and stability at linear as well as non linear level give constraint equation,

$$\lambda_{BDN} > 0, \chi_1 > 0, \eta \geq 0; \lambda \geq \eta, \tag{84}$$

$$c_s^2(3\chi_4 - 4\eta) \geq 0, \tag{85}$$

$$\lambda\chi_1 + c_s^2\lambda\left(\chi_4 - \frac{4\eta}{3}\right) \geq c_s^2\lambda\chi_2 + \lambda\chi_3 + \chi_2\chi_3 - \chi_1\left(\chi_4 - \frac{4}{3}\eta\right) \geq 0. \tag{86}$$

$$\zeta + \frac{4\eta}{3} \geq 0, \tag{87}$$

$$\begin{aligned} &3c_s^2\{\chi_1\left[\lambda^2\left(4\eta - 3\chi_4\right) + 3\chi_3\left(-\lambda^2 + \lambda\chi_2 + \chi_2^2\right)\right] \\ &+ \lambda[\lambda^2\left(4\eta + 3\chi_3 - 3\chi_4\right) + 3\chi_2^2\chi_3 + \lambda\chi_2\left(4\eta + 9\chi_3 - 3\chi_4\right)] \\ &+ \chi_1^2\left(4\eta - 3\chi_4\right)\left(2\lambda + \chi_2\right)\} - 9c_s^4\lambda^2\left(\chi_1 - \chi_2\right)\left(\lambda + \chi_2\right) \\ &+ \left(4\eta + 3\chi_3 - 3\chi_4\right)\left(\chi_1^2\left(4\eta - 3\chi_4\right) + 3\lambda\chi_3\left(\lambda + \chi_2\right) + 3\chi_2\chi_3\chi_1\right) \geq 0. \end{aligned} \tag{88}$$

- The entropy production using the more general entropy four-current is [20],

$$
\begin{aligned}
\partial_\mu S^\mu = & -\beta\Pi\Big[\theta+\beta_0\dot{\Pi}+\beta_{\Pi\Pi}\Pi\theta+\alpha_0\nabla_\mu n^\mu+\psi\alpha_{n\Pi}n_\mu\dot{u}^\mu+\psi\alpha_{\Pi n}n_\mu\nabla^\mu\alpha\Big] \\
& -\beta n^\mu\Big[T\nabla_\mu\alpha-\beta_1\dot{n}_\mu-\beta_{nn}n_\mu\theta+\alpha_0\nabla_\mu\Pi+\alpha_1\nabla_\nu\pi^\nu_\mu+\tilde{\psi}\alpha_{n\Pi}\Pi\dot{u}_\mu \\
& +\tilde{\psi}\alpha_{\Pi n}\Pi\nabla_\mu\alpha+\tilde{\chi}\alpha_{\pi n}\pi^\nu_\mu\nabla_\nu\alpha+\tilde{\chi}\alpha_{n\pi}\pi^\nu_\mu\dot{u}_\nu\Big] \\
& +\beta\pi^{\mu\nu}\Big[\sigma_{\mu\nu}-\beta_2\dot{\pi}_{\mu\nu}-\beta_{\pi\pi}\theta\pi_{\mu\nu}-\alpha_1\nabla_{\langle\mu}n_{\nu\rangle}-\chi\alpha_{\pi n}n_{\langle\mu}\nabla_{\nu\rangle}\alpha-\chi\alpha_{n\pi}n_{\langle\mu}\dot{u}_{\nu\rangle}\Big].
\end{aligned}
\tag{89}
$$

[20] W. Israel and J. M. Stewart, "Transient relativistic thermodynamics and kinetic theory," Annals Phys.118, 341 (1979).

- Second law of thermodynamics is satisfied if we impose the following relationships,

$$\Pi = -\zeta\Big[\theta + \beta_0\dot{\Pi} + \beta_{\Pi\Pi}\Pi\theta + \alpha_0\nabla_\mu n^\mu + \psi\alpha_{n\Pi}n_\mu\dot{u}^\mu + \psi\alpha_{\Pi n}n_\mu\nabla^\mu\alpha\Big] \tag{90}$$

$$\begin{aligned} n^\mu = \frac{\kappa}{T}\Bigg[& T\nabla_\mu\alpha - \beta_1\dot{n}_\mu - \beta_{nn}n_\mu\theta + \alpha_0\nabla_\mu\Pi + \alpha_1\nabla_\nu\pi^\nu_\mu + \tilde{\psi}\alpha_{n\Pi}\Pi\dot{u}_\mu \\ & + \tilde{\psi}\alpha_{\Pi n}\Pi\nabla_\mu\alpha + \tilde{\chi}\alpha_{\pi n}\pi^\nu_\mu\nabla_\nu\alpha + \tilde{\chi}\alpha_{n\pi}\pi^\nu_\mu\dot{u}_\nu\Bigg], \end{aligned} \tag{91}$$

$$\pi^{\mu\nu} = 2\eta\Big[\sigma_{\mu\nu} - \beta_2\dot{\pi}_{\mu\nu} - \beta_{\pi\pi}\theta\pi_{\mu\nu} - \alpha_1\nabla_{\langle\mu}n_{\nu\rangle} - \chi\alpha_{\pi n}n_{\langle\mu}\nabla_{\nu\rangle}\alpha - \chi\alpha_{n\pi}n_{\langle\mu}\dot{u}_{\nu\rangle}\Big]. \tag{92}$$

- The relativistic Navier-Stokes theory can then be understood to be valid only up to first order in the dissipative currents.

- It is important to note that in equilibrium following equations hold,

$$\dot{T} = 0;\ \dot{\mu} = 0;\ T\dot{u}^{\mu} + \Delta^{\mu\nu}\partial_{\nu}T = 0;\ E^{\alpha} - T\Delta^{\alpha\nu}\partial_{\nu}(\mu/T) = 0;$$
$$\partial_{\mu}u^{\mu} = 0, \sigma^{\alpha\beta} = 0, \tag{93}$$

- Further in equilibrium,

$$Q^{\mu} = 0,$$
$$\implies \theta_1\dot{u}^{\mu} + \frac{\theta_2}{T}\Delta^{\mu\lambda}\partial_{\lambda}T + \theta_3\Delta^{\mu\lambda}\partial_{\lambda}(\mu/T) = 0. \tag{94}$$

In the absence of external field $T\Delta^{\alpha\nu}\partial_{\nu}(\mu/T) = 0$,

$$\theta_1\dot{u}^{\mu} + \frac{\theta_2}{T}\Delta^{\mu\lambda}\partial_{\lambda}T = 0. \tag{95}$$

But in equilibrium, $T\dot{u}^{\mu} + \Delta^{\mu\nu}\partial_{\nu}T = 0 \implies \theta_1 = \theta_2$.

- Similarly in equilibrium $\mathcal{J}^{\mu} = 0$, and in the absence of external field, $T\Delta^{\alpha\nu}\partial_{\nu}(\mu/T) = 0$, thus,

$$\gamma_1\dot{u}^{\mu} + \frac{\gamma_2}{T}\Delta^{\mu\lambda}\partial_{\lambda}T = 0. \tag{96}$$

So for thermodynamic consistency we have $\gamma_1 = \gamma_2$. $\gamma_1 = \gamma_2$ along with $\theta_1 = \theta_2$ implies $l_1 = l_2$.

$$\mathcal{E} = T^{\mu\nu}u_\mu u_\nu; \ \ \mathcal{P} = \frac{1}{d}\Delta_{\mu\nu}T^{\mu\nu}; \ \ Q_\alpha = -\Delta_{\alpha\mu}u_\nu T^{\mu\nu}; \mathcal{N} = -u_\mu J^\mu;$$
$$\mathcal{T}_{\mu\nu} = \frac{1}{2}\left(\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} - \frac{2}{d}\Delta_{\mu\nu}\Delta_{\alpha\beta}\right)T^{\alpha\beta}; \mathcal{J}_\mu = \Delta_{\mu\alpha}J^\alpha. \tag{97}$$

- The decomposition of $T^{\mu\nu}$ and J^μ are just identities.
- Hydrodynamic picture through the derivative expansion of $\mathcal{E}$, $\mathcal{P}$, Q^μ, $\mathcal{T}^{\mu\nu}$, $\mathcal{N}$, and $\mathcal{J}^\mu$.
- To zeroth order in derivative expansion, there are two scalars, T and μ, no transverse vectors, and no transverse traceless 2-tensors.
- To first order in derivative expansion, there are three scalars, $u^\lambda\partial_\lambda T \equiv \dot{T}$, $\partial_\lambda u^\lambda$, and $u^\lambda\partial_\lambda\mu \equiv \dot{\mu}$.
- There are also three transverse vectors, $\Delta^{\rho\sigma}\partial_\sigma T$, $\dot{u}^\rho$, and $\Delta^{\rho\sigma}\partial_\sigma\mu$.
- There is one transverse traceless symmetric tensor, $\sigma^{\mu\nu} = \Delta^{\mu\rho}\Delta^{\nu\sigma}(\partial_\rho u_\sigma + \partial_\sigma u_\rho - \frac{2}{3}g_{\rho\sigma}\partial_\lambda u^\lambda)$.

- It turns out that relativistic generalization of the Navier-Stokes theory is unstable i.e. presence of the exponentially growing modes.
- It is also been argued that Navier-Stokes theory is acausal.
- Naively speaking the origin of the acausal nature is in the linear relations between dissipative currents and gradients of the primary fluid-dynamical variables.
- This imply that any inhomogeneity of α and u^{μ}, immediately results in dissipative currents.
- Israel and Stewart's formulation of causal relativistic dissipative fluid dynamics is the most popular and widely used.

- It is clear that one can go to a new frame (by choosing a_i and c_i appropriately) in which $\mathcal{E} = \epsilon$ and $\mathcal{N} = n$.

$$\varepsilon_i \to \varepsilon_i - \left(\frac{\partial \epsilon}{\partial T}\right) a_i - \left(\frac{\partial \epsilon}{\partial \mu}\right) c_i = 0, \tag{98}$$

$$\nu_i \to \nu_i - \left(\frac{\partial n}{\partial T}\right) a_i - \left(\frac{\partial n}{\partial \mu}\right) c_i = 0. \tag{99}$$

- In this frame,

$$\pi_i \to \pi_i - \left(\frac{\partial p}{\partial T}\right) a_i - \left(\frac{\partial p}{\partial \mu}\right) c_i = \pi_i - \varepsilon_i \left(\frac{\partial p}{\partial \epsilon}\right)_n - \nu_i \left(\frac{\partial p}{\partial n}\right)_\epsilon = f_i. \tag{100}$$

- If we choose $b_i = \frac{\theta_i}{\epsilon+p}$, then in the Landau frame $\theta_i = 0, \implies Q^\mu = 0$ and in this frame $\gamma_i \to \gamma_i - nb_i = \gamma_i - \frac{n}{\epsilon+p}\theta_i = l_i$.
- In the Landau frame,

$$T^{\mu\nu} = \epsilon u^{\mu\nu} + \left(p + f_1 \frac{\dot{T}}{T} + f_2 \partial.u + f_3 u^\lambda \partial_\lambda(\mu/T)\right) \Delta^{\mu\nu} - \eta\sigma^{\mu\nu} + \mathcal{O}(\partial^2), \tag{101}$$

$$J^\mu = nu^\mu + l_1 \dot{u}^\mu + \frac{l_2}{T}\Delta^{\mu\lambda}\partial_\lambda T + l_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + \mathcal{O}(\partial^2). \tag{102}$$

- For the type of Israel-Stewart theory considered here, causality and stability around equilibrium hold when $\eta/(s\tau_R T) \leq 1/2$ (where $s = 4\varepsilon/3T)$[21,22].
- The parameter λ does not appear contribute in a linearized analysis. So it can not be constrained.
- Kinetic theory (14-moment approximation) predicts the value of $\lambda$to be equal to 10/21, while the shear viscosity is given by $\eta = 4\varepsilon\tau_R/15$.
- One can show that causality in the FOCS theory is violated if one uses the 14-moment value $\lambda = 10/21$.
- Causality in the FOCS approach with coefficients given by (67)-(70) can be fulfilled, however, if λ is negative.
- The results presented herein may suggest that the IS parameter $\lambda = 10/21$ can be at odds with causality when one goes beyond the linearized regime.
- General statements about causality in the nonlinear regime of Israel-Stewart theory is not present in the presence of both shear and bulk viscosity.

[21] Denicol, G.S., Niemi, H., Molnar, E., Rischke, D.H..,arXiv:1202.4551.

[22] Pu, S., Koide, T., Rischke, D.H.. arXiv:0907.3906.