Soft singularities of two-loop QCD amplitudes with external massive quarks

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Białasówka, Kraków, 11 October 2024

Outline

1. Brief survey of fixed-order calculation techniques

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    - definition and relevance
    - details of the calculation
    - challenges and solutions thereof

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5. Conclusions

General situation

- Each collision at the LHC involves interactions of quarks and gluons
  ↪ **Understanding of strong interactions is critical to fully exploit potential of the LHC**
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Predictions in perturbative QCD

- In the region where the strong coupling $\alpha_s \ll 1$, fixed-order perturbative expansions is expected to work well

$$\sigma = \underbrace{\sigma_0}_{\text{LO}} + \underbrace{\alpha_s\sigma_1}_{\text{NLO}} + \underbrace{\alpha_s^2\sigma_2}_{\text{NNLO}} + \underbrace{\alpha_s^3\sigma_3}_{\text{N}^3\text{LO}} + \cdots$$

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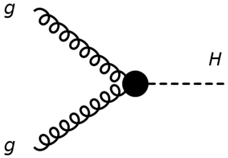
- N3LO

  [Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger, Baglio, Szafron, Chen, Gehrmann, Glover, Huss, Pelloni, Yang, Zhu '15 - '22]

$$pp \to H,\ Z/\gamma^*,\ W^\pm$$

Structure of perturbative QCD calculations

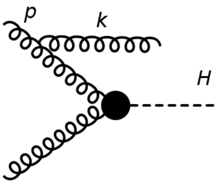
Leading Order (LO)



Structure of perturbative QCD calculations

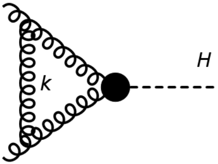
Next-to-Leading Order (NLO)

Real $\int d^4k$



$\Rightarrow$ divergent in the limits $k \to 0$ or $k \parallel p$ (implicit divergences)

Virtual $\int d^{4-2\epsilon}k$



$= \frac{1}{\epsilon^2} + \frac{1}{\epsilon} + \text{finite}$

Structure of perturbative QCD calculations

$$\sigma_{\mathrm{NLO}} = R + V$$

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How to carry out this cancellation in practice, given that R is integrated in 4 while V in d dimensions?

Structure of perturbative QCD calculations

- Subtraction

$$d = 4 - 2\epsilon$$

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$$= \underbrace{\int d^d k\ (R + V)\,\Theta(\chi_{\text{cut}} - \chi(k))}_{\text{unresolved}} + \underbrace{\int d^4 k\, R\,\Theta(\chi(k) - \chi_{\text{cut}})}_{\text{resolved}}$$

Kinematic regions of gluon emissions

Gluons' momenta in light-cone coordinates

$$k_i^\mu = \left(k_i^+, k_i^-, \boldsymbol{k}_i^\perp\right) \qquad \text{where} \qquad k^\pm = k^0 \pm k^3$$

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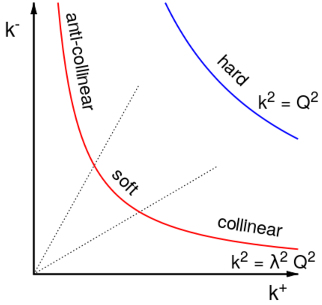
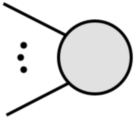
collinear	$k_i^\mu \sim (1, \lambda^2, \lambda)\, Q^2$
anti-collinear	$k_i^\mu \sim (\lambda^2, 1, \lambda)\, Q^2$
hard	$k_i^\mu \sim (1, 1, 1)\, Q^2$
soft	$k_i^\mu \sim (\lambda, \lambda, \lambda)\, Q^2$



where $\lambda \ll 1$ and $Q^2 \sim \mathcal{O}(1)$

Building blocks of N3LO amplitudes

- Born level



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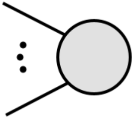

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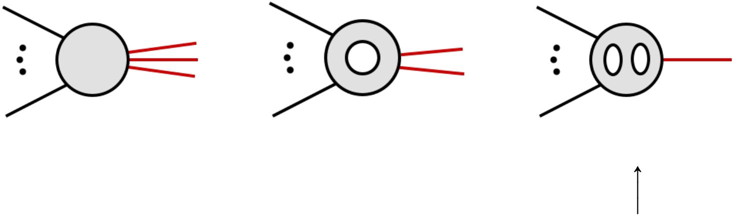
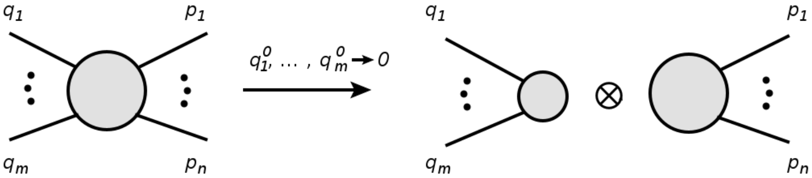


single soft limit at two loops

Soft factorization in QCD: tree level



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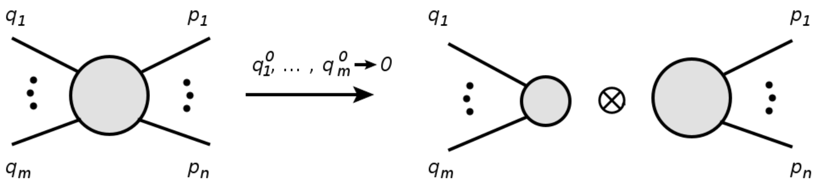

$$|\mathcal{M}^{(0)}(q_1,\ldots,q_m,p_1,\ldots,p_n)\rangle \xrightarrow{q_1^0,\ldots,q_m^0\to 0} \boldsymbol{J}^{(0)}(q_1,\ldots,q_m)\,|\mathcal{M}^{(0)}(p_1\ldots,p_n)\rangle$$

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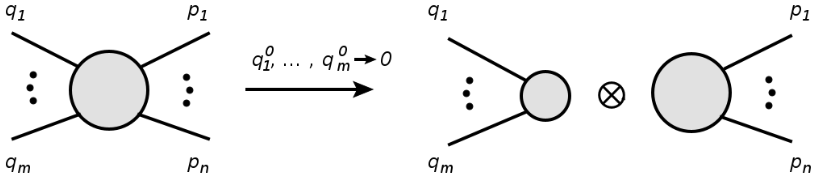

$$|\mathcal{M}^{(0)}(q_1,\ldots,q_m,p_1,\ldots,p_n)\rangle \overset{q_1^0,\ldots,q_m^0\to 0}{\longrightarrow} \boldsymbol{J}^{(0)}(q_1,\ldots,q_m)\,|\mathcal{M}^{(0)}(p_1\ldots,p_n)\rangle$$

- $\boldsymbol{J}^{(0)}(q_1,\ldots,q_m)$ is the soft current at tree level

Soft factorization in QCD: higher orders

One loop

$$|\mathcal{M}^{(1)}(q_1,\ldots,q_m,p_1,\ldots,p_n)\rangle \xrightarrow{q_1^0,\ldots,q_m^0\to 0} \boldsymbol{J}^{(1)}(q_1,\ldots,q_m)\,|\mathcal{M}^{(0)}(p_1\ldots,p_n)\rangle + \boldsymbol{J}^{(0)}(q_1,\ldots,q_m)\,|\mathcal{M}^{(1)}(p_1\ldots,p_n)\rangle$$

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Two loops

$$|\mathcal{M}^{(2)}(q_1,\ldots,q_m,p_1,\ldots,p_n)\rangle \xrightarrow{q_1^0,\ldots,q_m^0\to 0} \boldsymbol{J}^{(2)}(q_1,\ldots,q_m)\,|\mathcal{M}^{(0)}(p_1\ldots,p_n)\rangle + \boldsymbol{J}^{(1)}(q_1,\ldots,q_m)\,|\mathcal{M}^{(1)}(p_1\ldots,p_n)\rangle + \boldsymbol{J}^{(0)}(q_1,\ldots,q_m)\,|\mathcal{M}^{(2)}(p_1\ldots,p_n)\rangle$$

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The tree level result, for massive ($p_i^2 > 0$) and massless ($p_i^2 = 0$) hard partons, takes the simple form

$$\boldsymbol{J}_a^{\mu(0)} = \sum_{i=1}^{n} \boldsymbol{T}_i^a \frac{p_i^\mu}{p_i^\mu \cdot q} \,,$$

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$$\boldsymbol{J}_a^{\mu(1)} = \sum_{i=1}^{n} \boldsymbol{T}_i^a \boldsymbol{T}_j^b S(p_i, p_j, \{q_m\}) ,$$

while at two loops both from dipole and tripole emissions

$$\boldsymbol{J}_a^{\mu(1)} = \sum_{i \neq j} \boldsymbol{T}_i^{a_i} \boldsymbol{T}_j^{a_j} S_{ij}(p_i, p_j, \{q_m\}) + \sum_{i \neq j} \boldsymbol{T}_i^{a_i} \boldsymbol{T}_j^{a_j} \boldsymbol{T}_k^{a_k} S_{ijk}(p_i, p_j, p_k, \{q_m\}) .$$

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Massive fermions

- one loop
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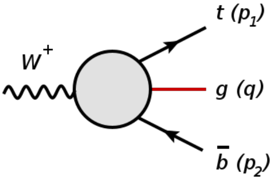
Our aim is to get the massive soft current at two loops to $\mathcal{O}(\epsilon)$

Kinematics



- Five invariants:

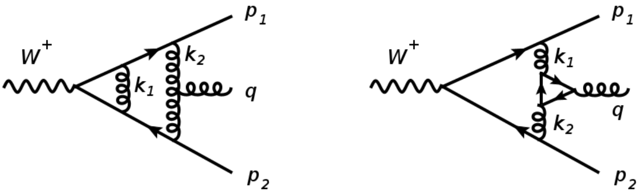
$$
\begin{aligned}
s_{1q} &= (p_1 + q)^2 \\
s_{2q} &= (p_2 + q)^2 \\
s_{12} &= (p_1 + p_2)^2 \\
m_t^2 &= p_1^2 \\
m_b^2 &= p_2^2
\end{aligned}
$$

Details of the calculation

- Generate two-loop diagrams (196 in total) for the process:

$$W^+ \to t + \bar{b} + g$$

in Feynman gauge, with FeynArts



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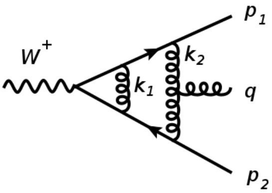

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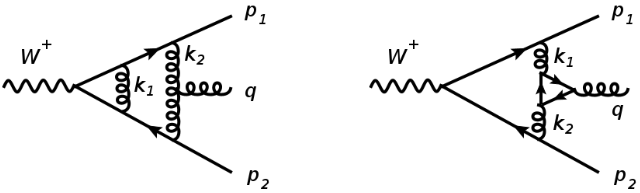
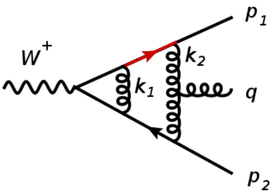


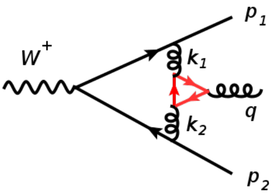
- Generate corresponding amplitude $\mathcal{A}^{(2)}_{W^+ \to t\bar{b}g}$ with FeynCalc
- Parameterize the gluon momenta

$$k_1 \to \lambda k_1 \,,$$
$$k_2 \to \lambda k_2 \,,$$
$$q \to \lambda k_1 \,,$$

expand the amplitude in λ and take the leading (most singular) term.
This is the soft limit of $\mathcal{A}^{(2)}_{W^+ \to t\bar{b}g}$

What happened to the propagators (and vertices)?





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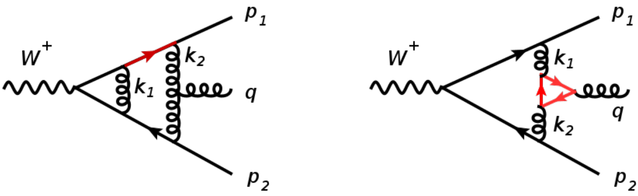

- Massive quarks:

$$\frac{\not{p}_1 - \lambda \not{k}_2}{(p_1 - \lambda k_2)^2 - m_t^2} = \frac{\not{p}_1 - \lambda \not{k}_2}{p_1^2 - \lambda p_1 \cdot k_2 + \lambda^2 k_2^2 - m_t^2} \simeq \frac{-\not{p}_1}{\lambda\, p_1 \cdot k_2} \quad \text{(eikonal)}$$

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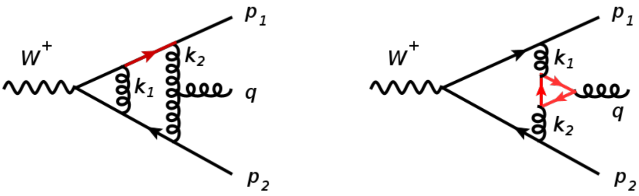

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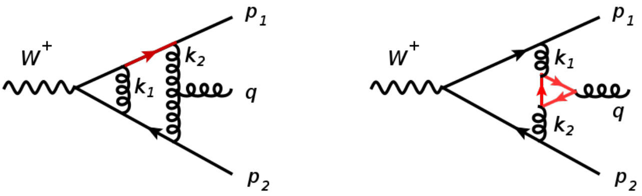

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- Gluons:

$$\frac{1}{(\lambda k_1)^2} = \frac{1}{\lambda^2 (k_1)^2} \quad \text{(exact)}$$

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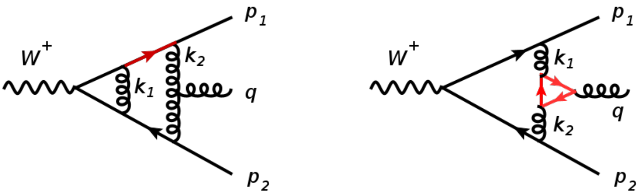

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$$\frac{\lambda \not{k}_1 - \lambda \not{k}_2}{(\lambda k_1 - \lambda k_2)^2} = \frac{\not{k}_1 - \not{k}_2}{\lambda (k_1 - k_2)^2} \quad \text{(exact)}$$

- Gluons:

$$\frac{1}{(\lambda k_1)^2} = \frac{1}{\lambda^2 (k_1)^2} \quad \text{(exact)}$$

- Tipple-gluon vertex: exact

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- We can significantly reduce the number of integrals by employing IBP identities

Integration by parts (IBPs)

In dimensional regularization, the integral over total derivative is zero

$$\int d^d k_1 \, \ldots d^d k_L \frac{\partial}{\partial k_i^\mu} \left(\frac{q^\mu}{P_1^{a_1} \cdots P_N^{a_N}} \right) = 0 \,,$$

where q is an arbitrary loop or external momentum.

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This generates a set of relations between the integrals

$$\sum_k c_{1,k} I_k = 0$$
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- For L loop momenta and E independent external momenta, we can built $L(L+E)$ relations.

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Example:

$$\mathrm{top}_1(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) =$$
$$\int \frac{d^d k_1 d^d k_2}{k_1^{2a_1} k_2^{2a_2} (k_1+k_2)^{2a_3} (k_2+q)^{2a_4} (k_1+k_2+q)^{2a_5} (2k_2 p_1)^{a_6} (2p_2(k_1+q))^{a_7} (-2k_1 p_1)^{a_8} (-2k_2 p_2)^{a_9}}$$

Differential equations method

We showed that IBP reduction allows us to write each integral in terms of a set of masters

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Which for multivariable case generalizes to

$$\frac{\partial}{\partial x_i} \vec{M} = \boldsymbol{a}_i\, \vec{M}$$

Variables

As mentioned earlier, the process is characterized by five invariants, or, equivalently, by five scalar products: $p_1 \cdot q, p_2 \cdot q, p_1 \cdot p_2, p_1^2, p_2^2$

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Hence, our integrals will also be invariant under rescaling of the heavy quark momenta, p_1 and p_2. This can be achieved only by the three ratios

$$\frac{(p_1 \cdot p_2)}{(p_1 \cdot 1)(p_2 \cdot q)}, \qquad \frac{(p_1 \cdot p_2)\,(p_2 \cdot q)}{(p_1 \cdot p_2)(p_1 \cdot q)}, \qquad \frac{(p_2 \cdot p_2)\,(p_1 \cdot q)}{(p_1 \cdot p_2)(p_2 \cdot q)}$$

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Loop integrals:

$$\int \prod dk_i^4 \to \text{ dimensionless}$$

$$\int \prod dk_i^{4-2\epsilon} \to \; m^{d-4} \text{ per loop}$$

Variables

Hence, our integrals will evaluate to the following functions:

$$I_i(p_1\cdot q, p_2\cdot q, p_1\cdot p_2, p_1^2, p_2^2) = q_\epsilon^{-2\epsilon} M_i(\alpha_1, \alpha_2)$$

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And we can rewrite our differential equations in terms of dimensionless functions M_i of dimensionless variables α_1, α_2:

$$\frac{\partial}{\partial \alpha_1} \vec{M}(\alpha_1, \alpha_2) = \boldsymbol{a}_1(\epsilon, \alpha_1, \alpha_2)\, \vec{M}(\alpha_1, \alpha_2)$$
$$\frac{\partial}{\partial \alpha_2} \vec{M}(\alpha_1, \alpha_2) = \boldsymbol{b}_2(\epsilon, \alpha_1, \alpha_2)\, \vec{M}(\alpha_1, \alpha_2)$$

System of differential equations

$$\frac{\partial}{\partial \alpha_1}\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{65} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,65} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,65} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,65} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{65,1} & a_{65,2} & a_{65,3} & \cdots & a_{65,65} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{65} \end{bmatrix}$$

$$\frac{\partial}{\partial \alpha_2}\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{65} \end{bmatrix} = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,65} \\ b_{2,1} & b_{2,2} & b_{2,3} & \cdots & b_{2,65} \\ b_{3,1} & b_{3,2} & b_{3,3} & \cdots & b_{3,65} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{65,1} & b_{65,2} & b_{65,3} & \cdots & b_{65,65} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{65} \end{bmatrix}$$

Closed subsystems

No.	Size homogeneous	Size inhomogeneous
1-14	1	1
15-26	2	1
27	2	2
28	2	2
29	3	1
30	3	1
31	4	2
32	4	2
33	5	1
34	5	2
35	5	1
36	5	2
37	5	1
38	6	2
39	6	2
40	8	4
41	10	1
42	10	1
43	12	2
44	13	1
45	13	1
46	16	2
47	29	3
48	29	3

Canonical form

All our differential systems, $s \in \{1, \ldots, 48\}$, have the form

$$\frac{\partial}{\partial \alpha_i} \vec{M}_s = \boldsymbol{A}_{si}(\alpha_i, \epsilon)\, \vec{M}_s$$

where $\vec{M}_s = \{M_1, \ldots, M_n\} \subset \{M_1, \ldots, M_{65}\}$.

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- The dependence on ϵ factorizes! This is the so-called canonical form.

Why is canonical form so useful?

Each integral in the equation

$$\partial_i \vec{J}_s = \epsilon\, \boldsymbol{S}_{si}(\alpha_i)\, \vec{J}_s\,, \quad \text{where} \quad \partial_i = \frac{\partial}{\partial \alpha_i}\,,$$

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has expansion in ϵ. Hence, we can write

$$\frac{\partial}{\partial \alpha_i}\left(J_s^{(0)} + J_s^{(1)}\epsilon + J_s^{(2)}\epsilon^2 + \ldots\right) = \epsilon \, \boldsymbol{S}_{si}\left(J_s^{(0)} + J_s^{(1)}\epsilon + J_s^{(2)}\epsilon^2 + \ldots\right)$$

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and we get the hierarchy of equations

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- The problem is essentially solved!

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is a rational function, *i.e.*

$$T_s^{jk}(x_i) = \frac{P(x_i)}{Q(x_i)} \,,$$

where P and Q are polynomials.

Letters and alphabet

Let's have a look at the canonical form again

$$\partial_i \vec{J} = \epsilon \, \boldsymbol{S}_i(\alpha_i) \, \vec{J}$$

Letters and alphabet

Let's have a look at the canonical form again

$$\partial_i \vec{J} = \epsilon\, \boldsymbol{S}_i(\alpha_i)\, \vec{J}$$

The entries of the matrix look as follows:

$$S_{i,mn} = \sum_j c_j^{mn}(\epsilon) \frac{1}{L_j(\alpha_i)}$$

where $L_j(\alpha_i)$ are letters of the alphabet.

Letters and alphabet

Let's have a look at the canonical form again

$$\partial_i \vec{J} = \epsilon \, \boldsymbol{S}_i(\alpha_i) \, \vec{J}$$

The entries of the matrix look as follows:

$$S_{i,mn} = \sum_j c_j^{mn}(\epsilon) \frac{1}{L_j(\alpha_i)}$$

where $L_j(\alpha_i)$ are letters of the alphabet.

In our problem, the alphabet consists of the following letters:

$$\{\alpha_1, \, \alpha_2, \, 1 - \alpha_1 - \alpha_2, \, 1 - 4\alpha_1\alpha_2\}$$

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Since, as shown earlier, the matrix $\boldsymbol{S}_i$ is being integrated iteratively, the integrals evaluate to multiple polylogarithms.

Multiple polylogarithms (MPLs)

$$G(c_1, \ldots, c_n; x) = \int_0^x \frac{dt}{t - c_1} G(c_2, \ldots, c_n; x)$$

$$G(; x) = 1$$

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Some special cases

$$G(0; x) = \log(x)$$
$$G(a; x) = \log\left(1 - \frac{x}{a}\right)$$
$$G(0, 1; x) = -\mathrm{Li}_2(x)$$

Square roots

A subset of our 48 systems can be put into canonical form directly by finding a rational transformation $\boldsymbol{T}$.

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However, the following change of kinematic variables

$$t_1 = 2\alpha_2, \quad t_2 = \sqrt{1-4\alpha_1\alpha_2}\,,$$

leading to a new alphabet

$$\{t_1,\ t_2,\ 1-t_1,\ 1-t_2,\ 1-t_1-t_2,\ 1-t_1+t_2\}\ ,$$

allowed us to find the canonical form for those cases.

So let's see what we have got

No.	Size homogeneous	Size inhomogeneous	Canonical form
1-14	1	1	✓
15-26	2	1	✓
27	2	2	✓
28	2	2	✓
29	3	1	✓
30	3	1	✓
31	4	2	✓
32	4	2	✓
33	5	1	✓
34	5	2	✓
35	5	1	✓
36	5	2	✓
37	5	1	✓
38	6	2	✓
39	6	2	✓
40	8	4	✓
41	10	1	✓
42	10	1	✓
43	12	2	X
44	13	1	✓
45	13	1	✓
46	16	2	X
47	29	3	✓
48	29	3	✓

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Let's put the systems 43 and 46 aside for a moment, and proceed with the remaining ones, which we have in the canonical form.

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$$J^{(0)}(x_1, x_2) = B^{(0)}$$

$$J^{(i)}(x_1, x_2) = \int_{(a_1,a_2)}^{(x_1,x_2)} (\boldsymbol{S}_1 dx_1' + \boldsymbol{S}_2 dx_2') J^{(i-1)}(x_1', x_2') + B^{(i)}$$

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  $\hookrightarrow$ AMFlow [Liu, Ma, Wang '18-'23]

AMFlow can also be used to numerically compute $J^{(i)}(x_1, x_2)$ outside of the boundary and this can serve an ultimate validation of our solutions!

So let's see what we have got

No.	Size homogeneous	Size inhomogeneous	Canonical form	Solved and validated with AMFlow
1-14	1	1	✓	✓
15-26	2	1	✓	✓
27	2	2	✓	✓
28	2	2	✓	✓
29	3	1	✓	✓
30	3	1	✓	✓
31	4	2	✓	✓
32	4	2	✓	✓
33	5	1	✓	✓
34	5	2	✓	✓
35	5	1	✓	✓
36	5	2	✓	✓
37	5	1	✓	✓
38	6	2	✓	✓
39	6	2	✓	✓
40	8	4	✓	✓
41	10	1	✓	✓
42	10	1	✓	✓
43	12	2	X	
44	13	1	✓	
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- Written as homogeneous: 12×12 system

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$$\frac{\partial}{\partial \alpha_1}\begin{bmatrix} M_{44} \\ M_{61} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} M_{44} \\ M_{61} \end{bmatrix} + \begin{bmatrix} R_{1,1} \\ R_{1,2} \end{bmatrix}$$

$$\frac{\partial}{\partial \alpha_2}\begin{bmatrix} M_{44} \\ M_{61} \end{bmatrix} = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \begin{bmatrix} M_{44} \\ M_{61} \end{bmatrix} + \begin{bmatrix} R_{2,1} \\ R_{2,2} \end{bmatrix}$$

where $R_{i,j}$ are given by the known functions:
$M_1, M_2, M_{15}, M_{18}, M_{20}, M_{26}, M_{32}, M_{53}, M_{54}, M_{55}$

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Let's focus on the homogeneous part of the 2×2 system.

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We already know that canonical form cannot be achieved with rational transformation in this case, even after change of variables.

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We already know that canonical form cannot be achieved with rational transformation in this case, even after change of variables.

But the above system of four equations can be written as two PDEs

$$r_1(x_1,x_2)\partial_1 M_{44} + r_2(x_1,x_2)\partial_2 M_{44} + r_3(x_1,x_2) M_{44} = 0$$
$$q_1(x_1,x_2)\partial_1 M_{61} + q_2(x_1,x_2)\partial_2 M_{61} + q_3(x_1,x_2) M_{61} = 0$$

System 43

The PDEs can be solved by the method of characteristics and we get

$$M^h_{44}(t_1, t_2) = t_1^{-3+6\epsilon} t_2^{-1+2\epsilon} (1 - t_1^4)^{-\epsilon} g_1 \left(\frac{1 - t_1^4 + t_2^2}{t_1^2 t_2^2} \right)$$

$$M^h_{61}(t_1, t_2) = t_1^{-3+6\epsilon} t_2^{-1+2\epsilon} (1 - t_1^4)^{-\epsilon} g_2 \left(\frac{1 - t_1^4 + t_2^2}{t_1^2 t_2^2} \right)$$

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Plugging this back to the original system gives a system of two ODEs

$$\frac{d}{dx}\begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} = \begin{bmatrix} \frac{(2\epsilon-1)}{2}\frac{x}{1-x^2} & \frac{(2\epsilon-1)}{2}\frac{1}{1-x^2} \\ \frac{(6\epsilon-1)}{2}\frac{1}{1-x^2} & \frac{(6\epsilon-1)}{2}\frac{x}{1-x^2} \end{bmatrix}\begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}$$

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Hence, a 2-variable problem reduces to a 1-variable problem.

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Hence, a 2-variable problem reduces to a 1-variable problem.

The question is: can we find a canonical form of the above matrix?

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- We could however try to find it manually!

System 43

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By definition, canonical form is achieved though the following transformation

$$\epsilon\, \boldsymbol{S} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} - \boldsymbol{T}^{-1}d\,\boldsymbol{T} \qquad (❀)$$

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where $\boldsymbol{A}$ is our original matrix and $\boldsymbol{T}$ is the transformation matrix we are looking for

$$\boldsymbol{T} = \begin{bmatrix} v_{11}(x) & v_{12}(x) \\ v_{21}(x) & v_{22}(x) \end{bmatrix}$$

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$$\left(\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} - \boldsymbol{T}^{-1}d\boldsymbol{T}\right)\Big|_{\epsilon=0} = 0$$

This leads to the following two systems

$$\frac{d}{dx}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} -\frac{x}{2(1-x^2)} & \frac{1}{2(1-x^2)} \\ -\frac{1}{2(1-x^2)} & \frac{x}{2(1-x^2)} \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}, \quad \frac{d}{dx}\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} -\frac{x}{2(1-x^2)} & \frac{1}{2(1-x^2)} \\ -\frac{1}{2(1-x^2)} & \frac{x}{2(1-x^2)} \end{bmatrix}\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}$$

System 43

$$\boldsymbol{T} = \frac{1}{2(c_2c_3 - c_1c_4)} \begin{bmatrix} c_3 P_{-\frac{1}{2}}(x) + c_4 Q_{-\frac{1}{2}}(x) & c_3 P_{\frac{1}{2}}(x) + c_4 Q_{\frac{1}{2}}(x) \\ c_1 P_{-\frac{1}{2}}(x) + c_2 Q_{-\frac{1}{2}}(x) & c_1 P_{\frac{1}{2}}(x) + c_2 Q_{\frac{1}{2}}(x) \end{bmatrix},$$

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where $P_n(x)$ and $Q_n(x)$ are Legendre polynomials, which can also be expressed via elliptic integrals:

$$P_{\frac{1}{2}}(x) = \frac{2}{\pi}\left[2E\left(\frac{1-x}{2}\right) - K\left(\frac{1-x}{2}\right)\right]$$

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- Hence, we found transformation to canonical form! We checked that it's invertible and it works.
- The transformation is not rational and it involves elliptic integrals.

Elliptic integrals

[Fagnano, Euler c. 1750]

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

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- Elliptic integrals are known to appear in more complicated two-loop calculations, especially when masses are involved.
- We managed to integrate the homogeneous part of the (g_1, g_2) system and found solutions which are combinations of elliptic functions E and K and polylogarithms.
- Our case is an explicit illustration of the fact that canonical form is not restricted to polylogarithms and it can be found also for cases with elliptic solutions.

State of the art

No.	Size homogeneous	Size inhomogeneous	Canonical form	Solved and validated with AMFlow
1-14	1	1	✓	✓
15-26	2	1	✓	✓
27	2	2	✓	✓
28	2	2	✓	✓
29	3	1	✓	✓
30	3	1	✓	✓
31	4	2	✓	✓
32	4	2	✓	✓
33	5	1	✓	✓
34	5	2	✓	✓
35	5	1	✓	✓
36	5	2	✓	✓
37	5	1	✓	✓
38	6	2	✓	✓
39	6	2	✓	✓
40	8	4	✓	✓
41	10	1	✓	✓
42	10	1	✓	✓
43	12	2	X	
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45	13	1	✓	
46	16	2	X	
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39	6	2	✓	✓
40	8	4	✓	✓
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- Our calculation is well advanced. We computed and validated majority of master integrals.
- We encountered several integrals which cannot be expressed through MPLs but involve also elliptic integrals.
- This is not surprising for a two-loop calculation with massive particles.
- Work in progress on the remaining integrals but all conceptual problems seem to be solved.