Entanglement entropy, Krylov complexity and Deep inelastic scattering data

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Based on:

Based on

Eur.Phys.J.C 82 (2022) 2, 111
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2814098
M. Hentschinski, D. Kharzeev, K. Kutak, Z. Tu

2404.07657
P. Caputa, K. Kutak

My motivation

Bounds and properties of EE may provide some new insight on behavior of parton density functions

Links to other areas (thermodynamics, gravity, quantum information, conformal field theory)

Interesting in context of parton saturation and thermalization problem of Quark Gluon Plasma

Boltzman and von Neuman entropy formulas – reminder

The entropy S of macrostate is given by the log of number W of distinct microstates that compose it

$$S = -\sum_{i=1}^{W} p(i) \ln p(i)$$ Gibbs entropy

For uniform distribution $p(i) = \frac{1}{W}$ the entropy is maximal

Boltzmann entropy

$$S = \ln W$$

Since partons are introduced as the microscopic constituents that compose the macroscopic state of the proton, it is natural to evaluate the corresponding entropy or entropy corresponding to parton density.

K. Kutak '11, Peschanski'12
A. Kovner, M. Lublinsky '15
D. Kharzeev, E. Levin '17,...

But proton as a whole is a pure state and the von Neuman entropy is 0. Can one get any nontrivial result?

For pure state (one state) density matrix is:

$$\rho = |\psi\rangle\langle\psi|$$

$$S_{VN} = -Tr[\rho \ln \rho] = -1 \ln 1 = 0$$

For mixed state i.e. classical statistical mixture

$$\rho = \sum p(i) |\psi_i\rangle\langle\psi_i|$$

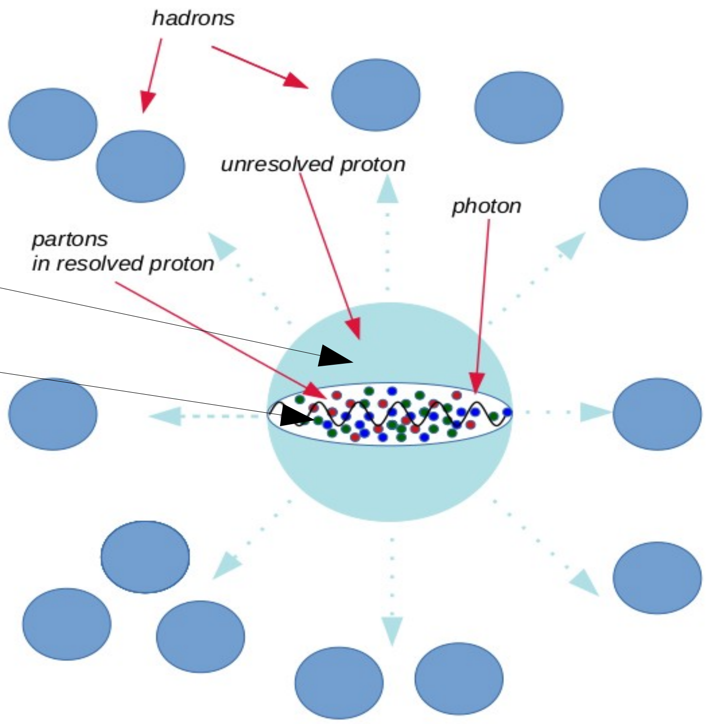
$$S_{VN} \neq 0$$

Kharzeev, Levin '17

Entanglement entropy in DIS

The composite system is described by

$|\Psi_{AB}\rangle$ in $A \cap B$



proton's rest frame

$\mathcal{H}_B$ of dimension n_B.

$\mathcal{H}_A$ of dimension n_A

Kharzeev, Levin '17

general definitions

entangled
if the product can not be expressed as separable product state

$$|\Psi_{AB}\rangle = \sum_{i,j} c_{ij} |\varphi_i^A\rangle \otimes |\varphi_j^B\rangle$$

separable
if the product can be expressed as separable product state

$$|\Psi_{AB}\rangle = |\varphi^A\rangle \otimes |\varphi^B\rangle$$

Schmidt decomposition

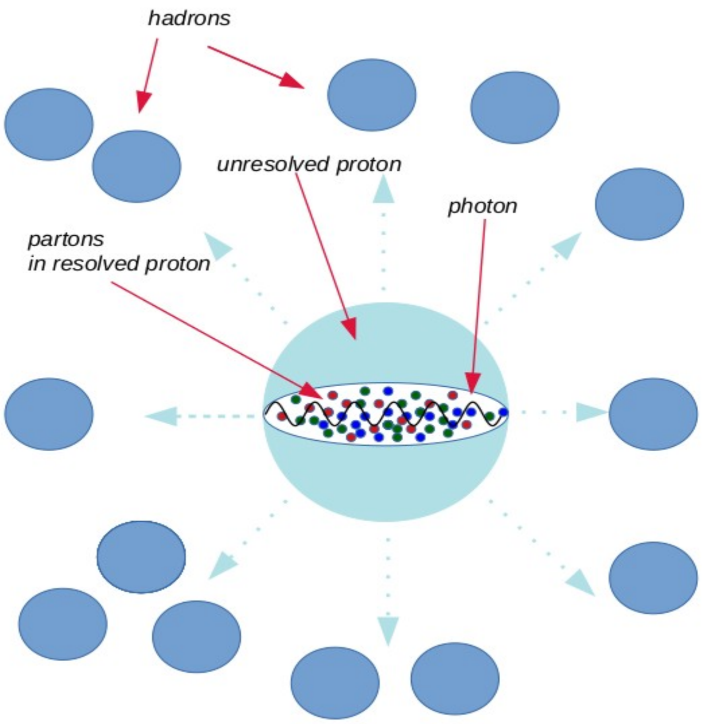
$$|\Psi_{AB}\rangle = \sum \alpha_n |\Psi_n^A\rangle |\Psi_n^B\rangle$$

orthonormal states belonging to A

orthonormal states bolonging to B

related to matrix C

Entanglement entropy in DIS



$$|\Psi_{AB}\rangle = \sum_n \alpha_n |\Psi_n^A\rangle |\Psi_n^B\rangle$$

$$\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$$

$$\rho_A = \mathrm{tr}_B \rho_{AB} = \sum_n \alpha_n^2 |\Psi_n^A\rangle\langle\Psi_n^A|$$

The density matrix of the mixed state probed in region A

$\alpha_n^2 \equiv p_n$ probability of state with n partons

Kharzeev, Levin '17

$$S = -\sum_n p_n \ln p_n$$

entropy results from the entanglement between the regions A and B, and can thus be interpreted as the entanglement entropy. Entropy of region A is the same as entropy in region B.

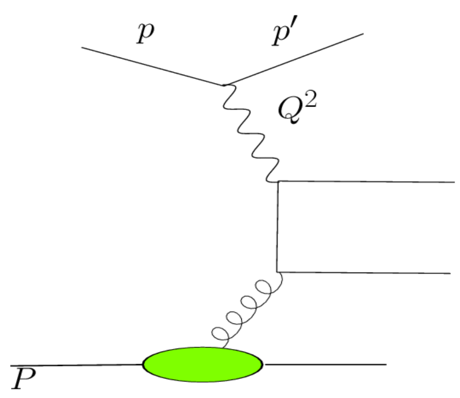
Proton structure function and dipole cross section

$$F_2(x,Q^2) = \frac{Q^2}{4\pi^2}\alpha_s \sum_q e_q^2 \int d^2k\, \mathcal{F}(x,k^2) \left(S_L(k^2,Q^2,m_q^2) + S_T(k^2,Q^2,m_q^2)\right)$$

dipole gluon density

impact factors ~ hard coefficients

In the kt factorization

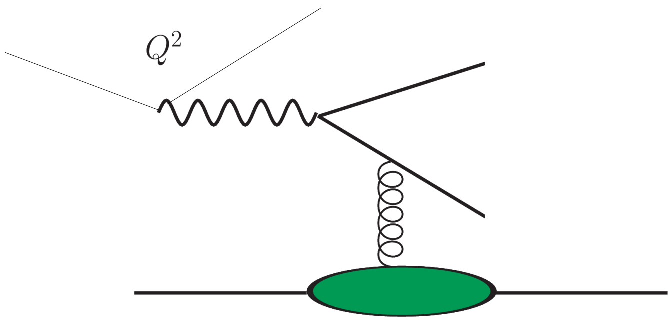


Boosted proton

$$\boxed{xg(x,Q) = \int_0^{Q^2} d\mathbf{k}^2 \mathcal{F}(x,\mathbf{k}^2)}$$

number of gluons at
“seen” at scale Q

In the dipole formalism



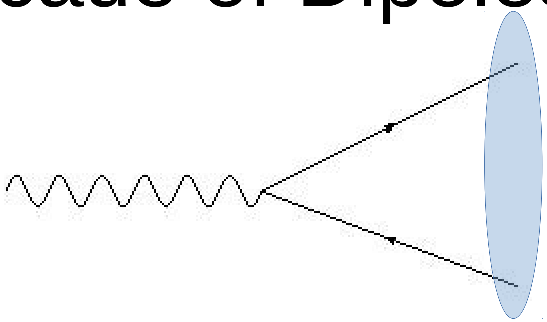
Proton in rest frame

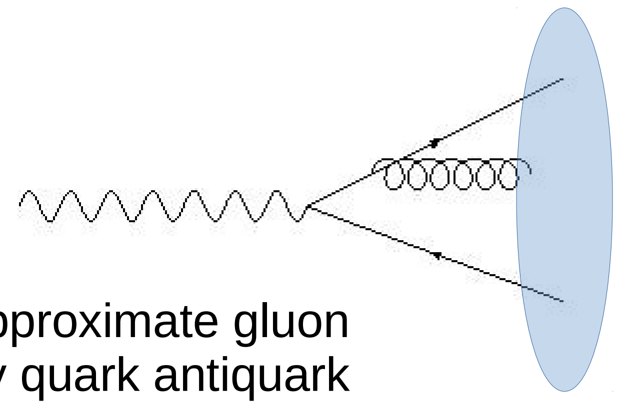
$$F_2(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{em}} \int d^2b \int_0^1 dz \int d^2r \left(|\psi_L(z,r)|^2 + |\psi_T(z,r)|^2\right) N(x,r,b)$$

wave function

dipole amplitude

Cascade of Dipoles



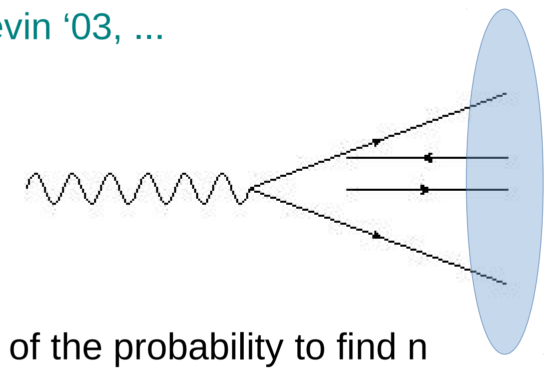


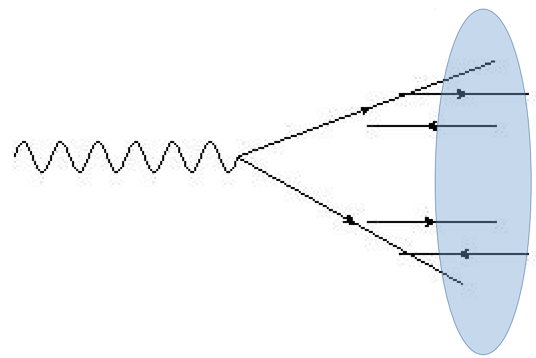
$$p_n = P_n$$

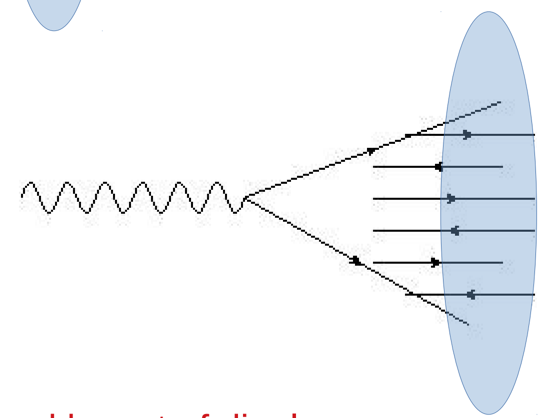
microstates – probabilities for given number of dipoles

approximate gluon by quark antiquark

Mueller 95,
Lublinsky, Levin '03, ...







depletion of the probability to find n dipoles due to the splitting into (n + 1) dipoles.

+ ...

$$\frac{dP_n(Y)}{dY} = -\lambda n P_n(Y) + (n-1)\lambda P_{n-1}(Y)$$

$$P_1(0) = 1$$

$$P_{n>1}(0) = 0$$

model of BFKL dipole cascade

growth due to the splitting
of (n − 1) dipoles into n dipoles.

set of partons is described by set of dipoles with fixed sizes ,Y is rapidity and is related to energy

depletion of the probability to find n dipoles due to the splitting into (n + 1) dipoles

the growth due to the splitting of (n − 1) dipoles into n dipoles

$$P_n(Y) = \frac{1}{n!}\int d^2r_1 d^2b_1 \cdots d^2r_n d^2b_n \frac{|\Psi^{[n]}(\vec{r}_{1\perp}, \vec{b}_{1\perp}, \ldots, \vec{r}_{n\perp}, \vec{b}_{n\perp}, Y)|^2}{\sum_{\sigma\sigma'} |\Psi^{(0)}_{\sigma\sigma'}(\vec{x}_{10}, z_1)|^2 \Big|_{O(\alpha_s^0)}},$$

Cascade of Dipoles – parton density and entropy

$$p_n(y) = \frac{e^{-\Delta y}}{C}\left(1 - \frac{e^{-\Delta y}}{C}\right)^{n-1}$$

constant

Kharzeev, Levin '17

BFKL intercept

$$\langle n \rangle_y = \sum_n n p_n(y) = C e^{\Delta y} \equiv \bar{n}(x)$$

mean number of dipoles,
interpreted as parton density

$$\bar{n}(x,\mu) \equiv \frac{dn}{d\ln 1/x} = x\Sigma(x,\mu) + xg(x)$$

$$S_{inc.}(\bar{n}) = -\sum_n p_n(\bar{n}) \ln p_n(\bar{n}) \quad = \ln \bar{n} - (\bar{n}-1)\ln\left(1 - \frac{1}{\bar{n}}\right)$$

$$S_{\mathrm{inc.}}^{\mathrm{univ.}}(\bar{n}) = \ln(\bar{n})$$

This is also formulated in 3+1 D

And for hard scale dependence case
Nowak, Liu, Zahed '22, Nowak, Liu, Zahed '23

Krylov subspace, complexity – motivation

Simple reference quantum state spreads and becomes complex in Hilbert space

Visvanath, Muller ‘63
Altman, Avdoshkin, Cao, Parker, Scaffidi ‘19
Balasubramanian, Caputa, Magan, Wu ‘22,...

$$i\partial_t |\Psi(t)\rangle = H |\Psi(t)\rangle \qquad |\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle$$

Krylov basis

$$|\Psi(t)\rangle = e^{-iHt} |\Psi_0\rangle = \sum_n \phi_n(t) |K_n\rangle \qquad p_n(t) = |\phi_n(t)|^2 \qquad \sum_n |\phi_n(t)|^2 = 1$$

Comes from studies of efficient diagonalization of matrices and computation of characteristic polynomial coefficients

The Krylov complexity used to quantify chaotic behavior of quantum systems.

$$\mathcal{C}_K(t) = \langle n \rangle = \sum_n n\, p_n(t)$$

One can get Lapunov exponent of considered system

Krylov subspace – construction

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} |\Psi_n\rangle$$

$$|\Psi_n\rangle \equiv \{|\Psi_0\rangle, H|\Psi_0\rangle, ..., H^n|\Psi_0\rangle, ...\}$$

n consequtive application
of Hamiltonian

$$|K_0\rangle = |\psi(0)\rangle = |\psi_0\rangle$$

Gram-Schmidt orthogonalization procedure. Construct with K_2 by subtracting the previous two vectors, K_3 by subtracting the previous 3 vectors, and so forth

$$|z_1\rangle = \hat{H}|K_0\rangle - a_0|K_0\rangle \qquad |K_1\rangle = \frac{|z_1\rangle}{\langle z_1|z_1\rangle}$$

$$|z_{n+1}\rangle = (\hat{H} - a_n)|K_n\rangle - b_n|K_{n-1}\rangle$$

Strength of the Lanczos algorithm. n + 1 is determined by n and n – 1. Low memory requirements Visvanath, Muller ‘63

$$|K_n\rangle = b_n^{-1}|z_n\rangle \qquad b_n = \langle z_n|z_n\rangle^{\frac{1}{2}} \qquad a_n = \langle K_n|\hat{H}|K_n\rangle$$

Project a high-dimensional problem
onto a lower-dimensional Krylov subspace

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle$$

$$\langle K_n|K_m\rangle = \delta_{nm}$$

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_n \phi_n(t)|K_n\rangle$$

$$H_{nm} := \langle K_n|\hat{H}|K_m\rangle = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots \\ b_1 & a_2 & b_2 & 0 & \cdots \\ 0 & b_2 & a_3 & b_3 & \cdots \\ 0 & 0 & b_3 & a_4 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

probability amplitudes
for each vector

$$i\partial_t\phi_n(t) = a_n\phi_n(t) + b_n\phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t)$$

Krylov basis and dipole evolution

P. Caputa, K. Kutak, 2404.07657

$$i\partial_t \phi_n(t) = a_n \phi_n(t) + b_n \phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t)$$

The equation above can be solved for

$$a_n = 0$$
$$b_n = \alpha n$$

$$\phi_n(t) = (-i)^n \frac{\tanh^n(\alpha t)}{\cosh(\alpha t)}$$

$$|\psi(t)\rangle = e^{-i\alpha(L_1 + L_{-1})t}\,|0\rangle \otimes |0\rangle$$

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \frac{\tanh^n(\alpha t)}{\cosh^{2h}(\alpha t)} \sqrt{\frac{\Gamma(2h+n)}{n!\Gamma(2h)}}\, |n\rangle \otimes |n\rangle$$

see also Braun, Vacca '06
Hamiltinian that they use
has the same symmetries

$$p_n(t) = |\phi_n(t)|^2$$

$$\mathcal{C}_K(t) = \sum^{\infty} n|\phi_n(t)|^2$$

excitation index

$$\rho(t) = \sum_{n=0}^{\infty} p_n(t)\,|n\rangle\langle n|$$

conformal weight

$$\mathcal{C}_K(t) = \sum_{n=0}^{\infty} n|\psi_n(t)|^2 = \sinh^2(\alpha t)$$

After expressing it in terms rapidity and
probabilities rapidity variable one gets

$$p_n(Y) = \frac{\Gamma(2h+n)}{n!\Gamma(2h)} (e^{-\alpha Y})^{2h} \left(1 - e^{-\alpha Y}\right)^n$$

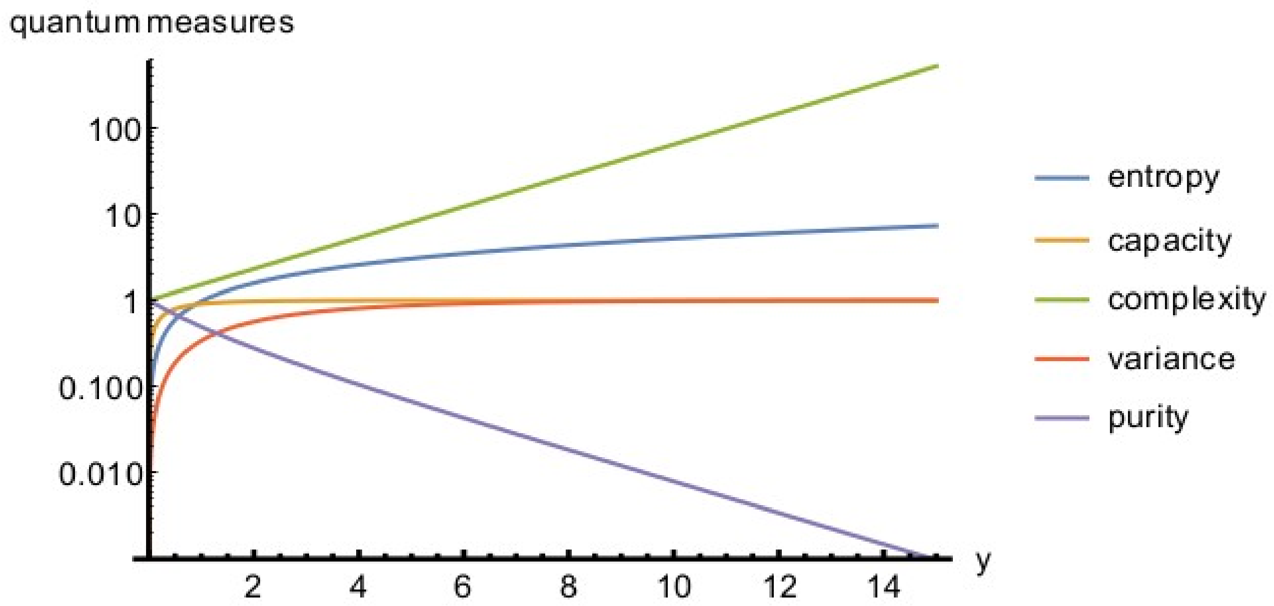
$$\mathcal{C}_K(Y) = e^{\alpha Y} - 1 \qquad Y = \ln(1/x)$$

$$\partial_Y p_n(Y) = \alpha n\, p_{n-1}(Y) - \alpha(n+1)\, p_n(Y)$$

$$\boxed{\mathcal{C}_K = xg(x)}$$

QI measures and dipole equations

P. Caputa, K. Kutak, 2404.07657



$$S_K(t) = -\sum_n p_n(t) \log p_n(t)$$

$$C_E = \lim_{m \to 1} m^2 \partial_m^2 [(1-m)\, S_K^{(m)}].$$

$$\mathcal{C}_K(t) = \langle n \rangle = \sum_n n\, p_n(t)$$

$$\delta_K^2 = \frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle^2}$$

$$\gamma_K = \sum_n p_n^2(t).$$

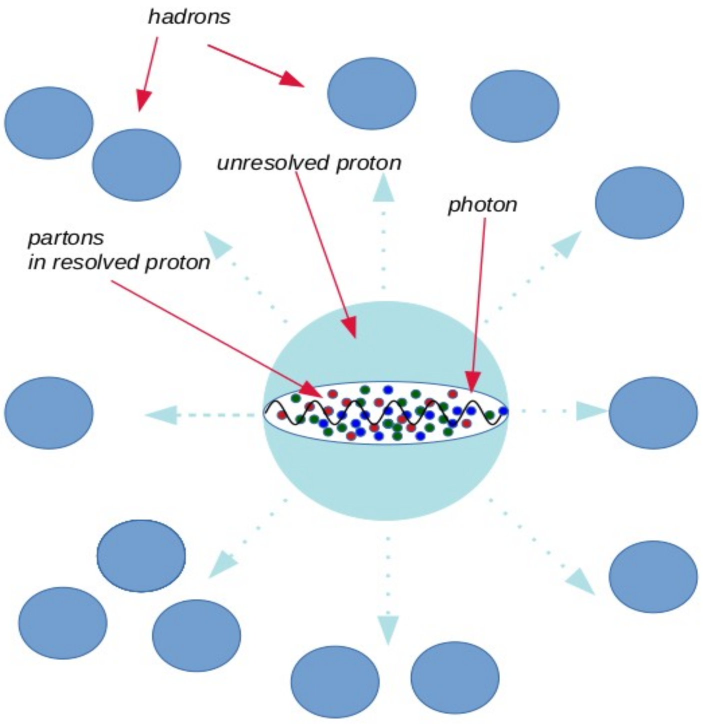
$$\partial_Y p_n(Y) = -\lambda n p_n(Y) + \lambda(n-1) p_{n-1}(Y)$$

Entanglement entropy – calculation and measurement

For DIS at high energies, this entanglement entropy can be calculated using

$$S(x, Q^2) = \ln \left\langle n \left(\ln \frac{1}{x}, Q \right) \right\rangle$$

Calculated using parton/dipole density



Conjecture that these entropies are the same

$$S_{hadron} = \sum P(N) \ln P(N)$$

Measured by counting hadrons

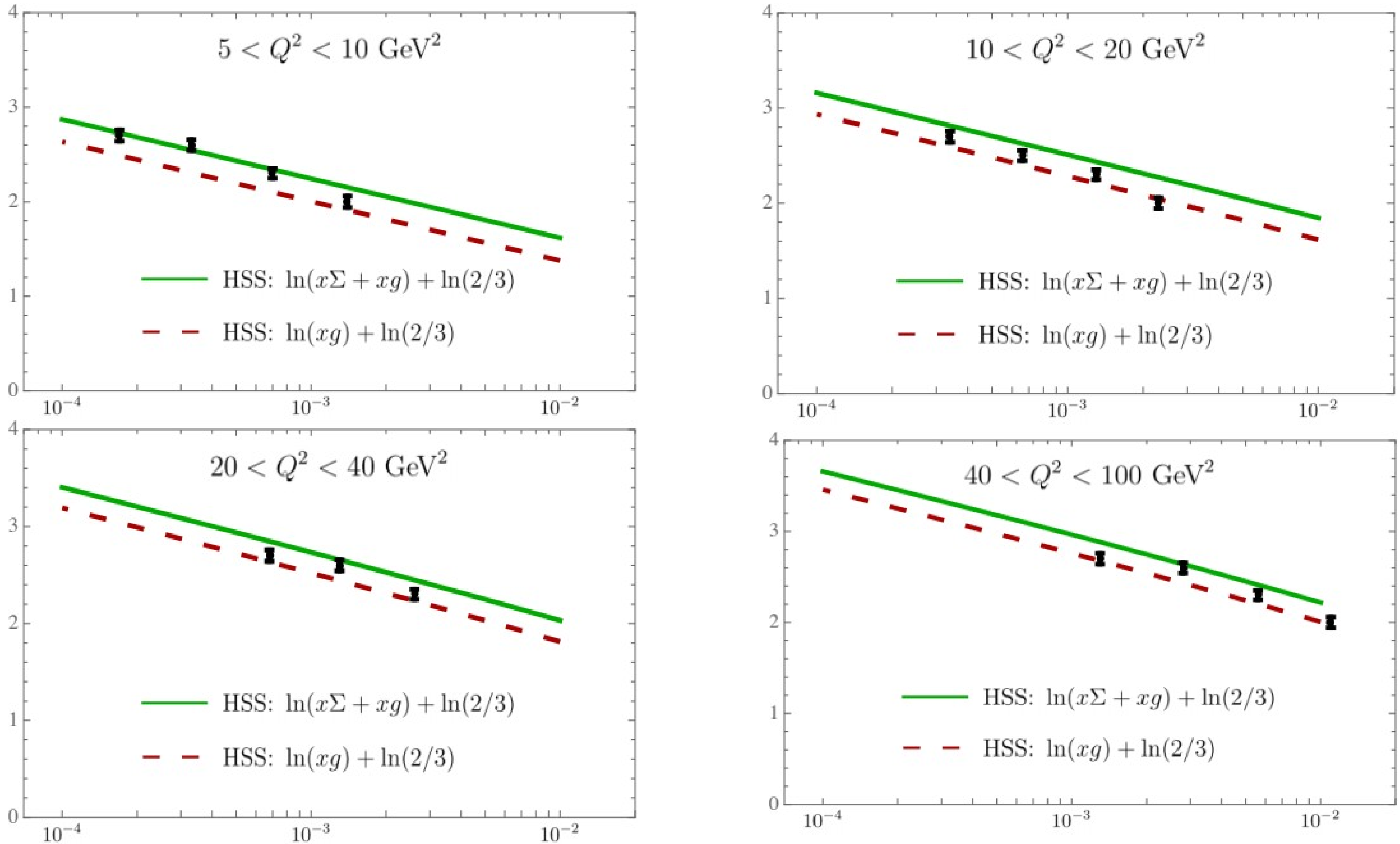
number of measured hadrons

The charged particle multiplicity distribution measured in either the current fragmentation region or the target fragmentation region.

Fraction of events with charged hadron

Results

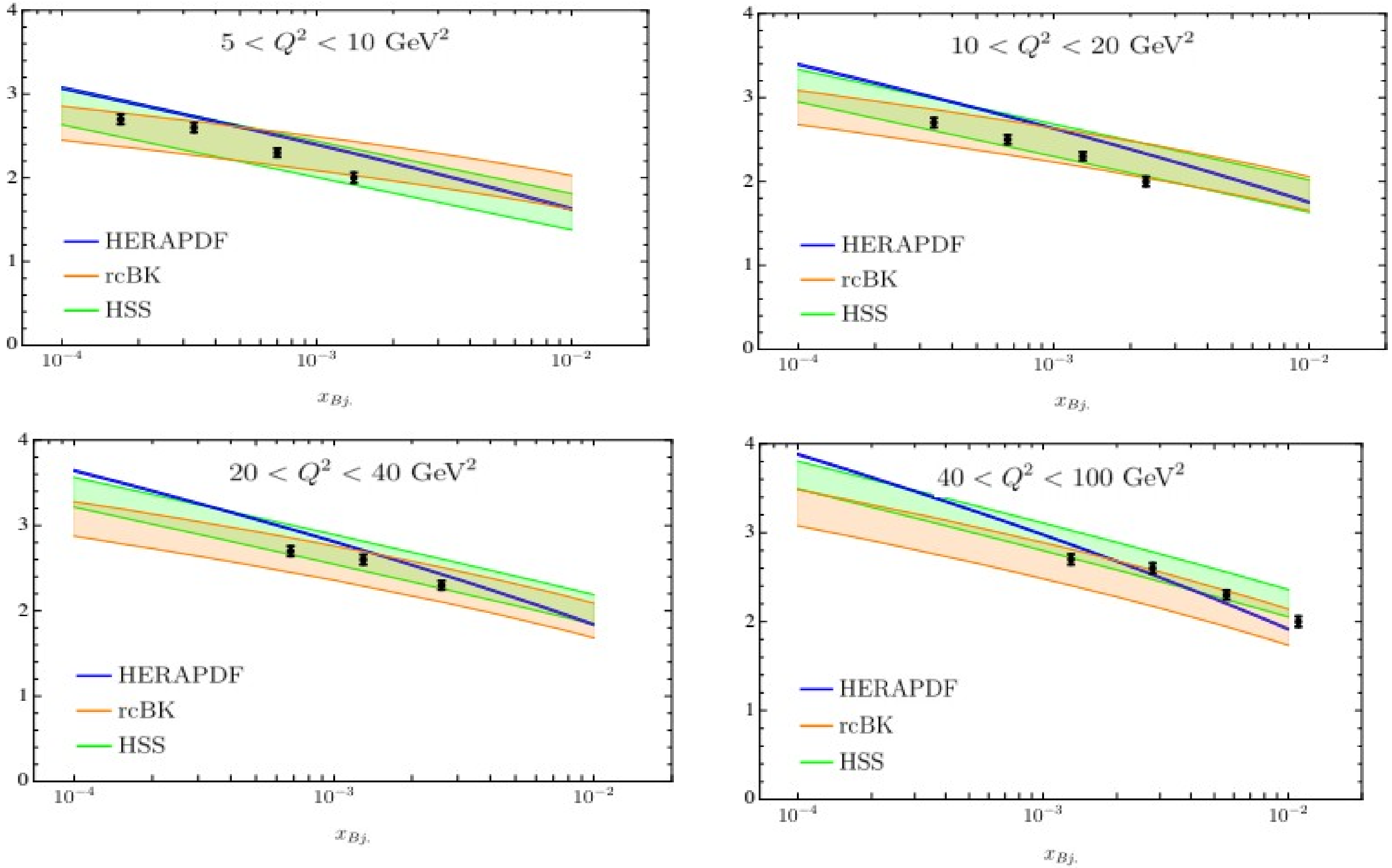
Eur.Phys.J.C 82 (2022) 2, 111 Hentschinski, Kutak



Hint that the general idea works. Gluon dominates over quarks.
One has to also take into account that only charged hadrons were measured.

Large scales - description

Martin Hentschinski, KK, Robert Straka ‘23



Entropy formula - interpretation

$$P_n(Y) = e^{-\lambda Y}\left(1 - e^{-\lambda Y}\right)^{n-1}$$

At low x partonic microstates have equal probabilities

In this equipartitioned state the entropy is maximal – the partonic state at small x is maximally entangled.

In terms of information theory as Shanon entropy:
- equipartitioning in the maximally entangled state means that all “signals” with different number of partons are equally likely
- it is impossible to predict how many partons will be detected in a give event.
- structure function at small x should become universal for all hadrons.

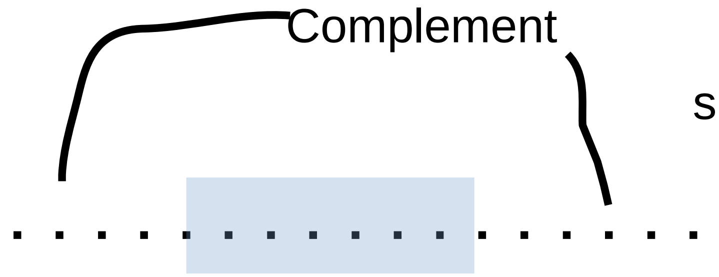
Comments

CFT result for EE

central charge

$$S = \frac{c}{3} \ln \frac{L}{\epsilon}$$

UV cutoff



spnis in 1 D

Region A of length L

Entanglement entropy obtained from CFT calculations as well as from gravity using Ryu-Takayanagi formula

Relation to Kharzeev-Levin formula

$$L = (mx)^{-1} \qquad \epsilon \equiv 1/m$$

Length of tube probed in DIS

Proton’s Compton wave length

$$S = \ln \left(\frac{1}{x} \right)^{1/3}$$

$$S(x) = \ln(xg(x))$$

See also
Callan, Wilczek ‘94
Calabrese, Cardy ‘04

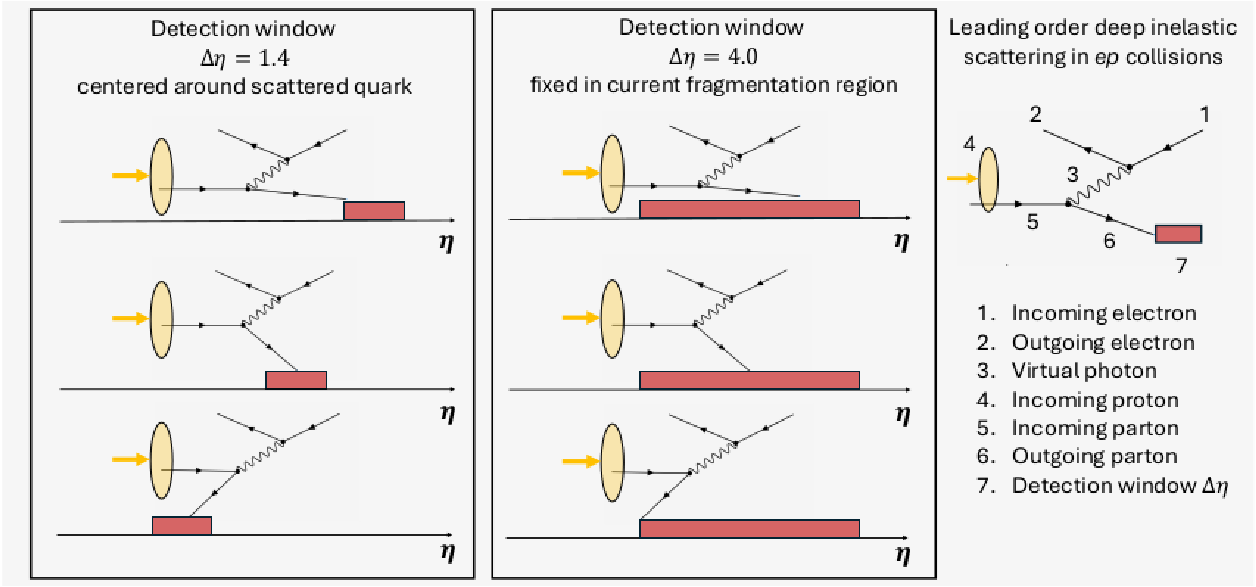
and lectures by
Headrick

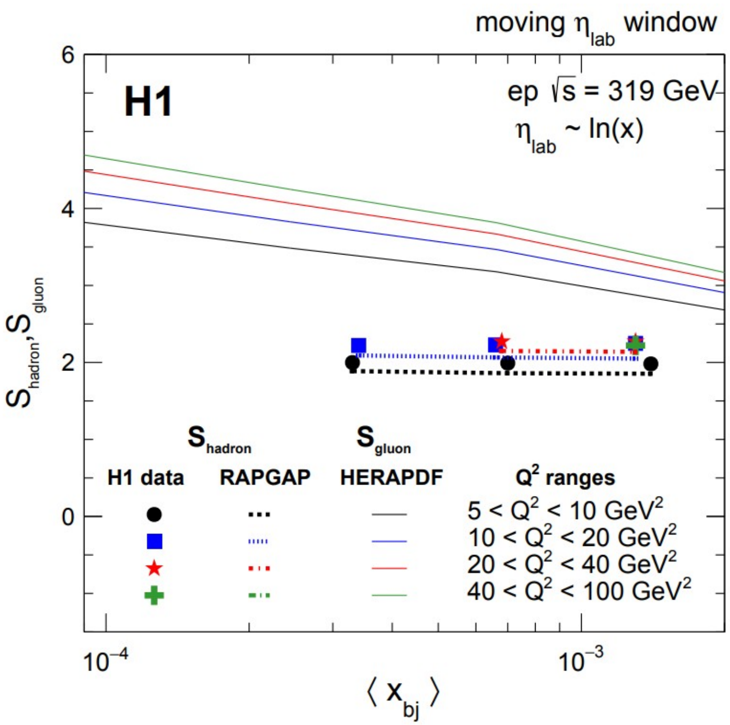
Studied also in the context of 2 D QCD

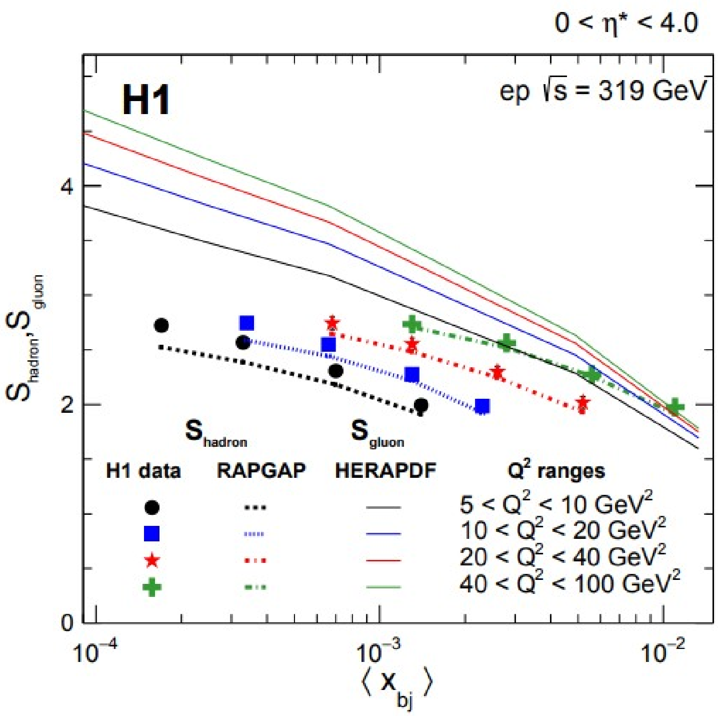
Liu, Nowak, Zahed, ‘22

Casini, Huerta, Hosco ‘05

Entropy measurements







Generalization of KL formula

Hentschinski, Kharzeev, Kutak, Tu ‘24

Generalize the KL formula to address measurement in rapidity window

$$p_n(y) = \sum_{m=0}^{n} p_{n-m}(y_0)\tilde{p}_m(y, y_0)$$

$$\sum_{m=0} \tilde{p}_m(y, y_0) = 1$$

probability to have m dipoles in $[y_0,y]$

n-m dipoles in the range $[0,y_0]$

probability to have n dipoles in [0,y]

$$\langle n \rangle_{y;y_0} = \sum n\tilde{p}_n(y, y_0)$$

$$\tilde{p}_0(y, y_0) = e^{-\Delta(y-y_0)}$$

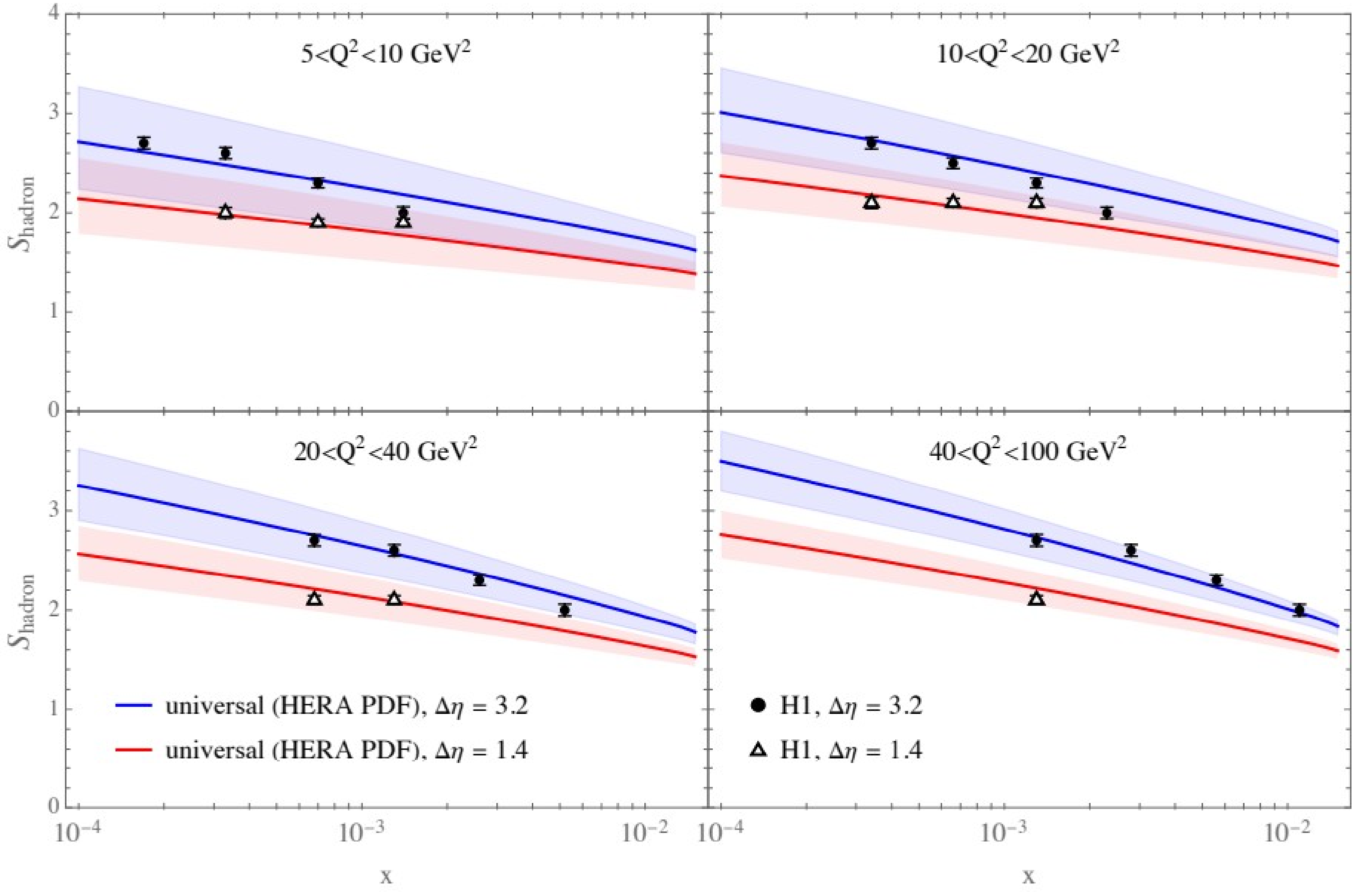
$$\tilde{p}_{n\geq 1}(y, y_0) = p_n(y) \cdot (1 - \tilde{p}_0(y, y_0))$$

$$S_{loc}(\bar{n}, \tilde{p}_0) = -\sum_{n=0} \tilde{p}_n(\bar{n}, \tilde{p}_0) \ln \tilde{p}_n(\bar{n}, \tilde{p}_0)$$

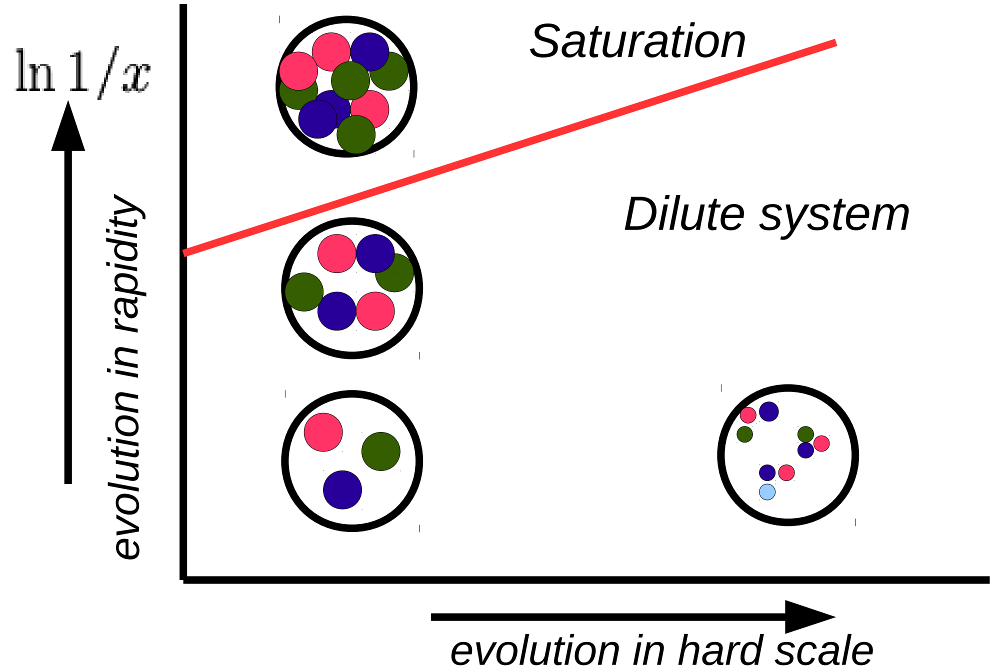
$$= -\tilde{p}_0 \ln \tilde{p}_0 - (1 - \tilde{p}_0) \ln(1 - \tilde{p}_0) + (1 - \tilde{p}_0) S_{inc.}(\bar{n})$$

For large rapidity window this formula reduces to $S_{\text{inc.}}^{\text{univ.}}(\bar{n}) = \ln(\bar{n})$

Data description



Gluons at high energies



Saturation – state where number of gluons stops growing due to high occupation number. Way to fulfill unitarity requirements in high energy limit of QCD.

L.V. Gribov, E.M. Levin, M.G. Ryskin
Phys.Rept. 100 (1983) 1-150

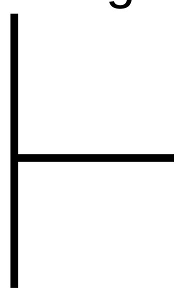
Larry D. McLerran, Raju Venugopalan
Phys.Rev. D49 (1994) 3352-3355

Phenomenological model:
Golec-Biernat, Wusthoff '99

On microscopic level it means that
gluon apart splitting recombine

splitting

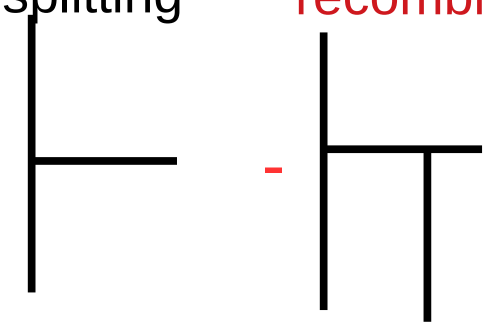
Linear evolution
Equation
BFKL



Nonlinear evolution equations
BK, JIMWLK
Balitcky-Kovchegov,

Jailian-Marian,Iancu
McLerran,Weigert,Leonidov,Kovner

splitting recombination



Bartels,
Wusthoff '93

QI measures and dipole equations

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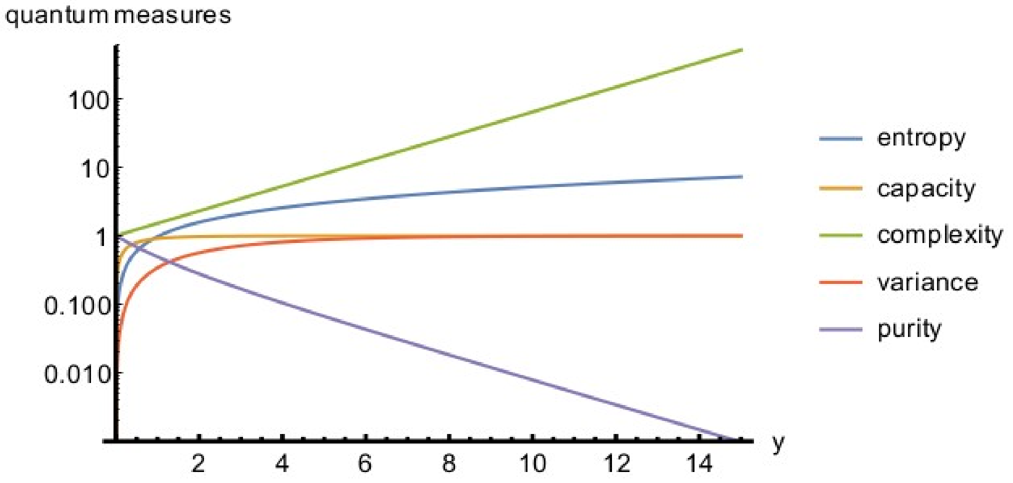
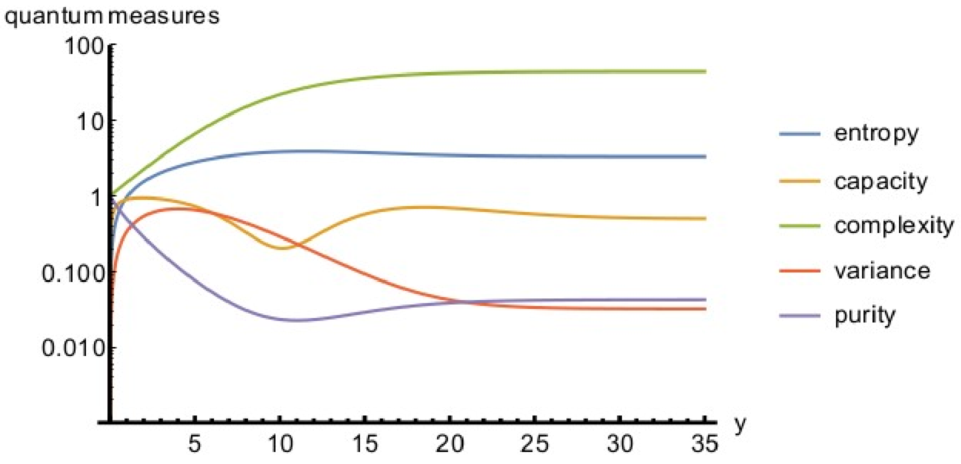




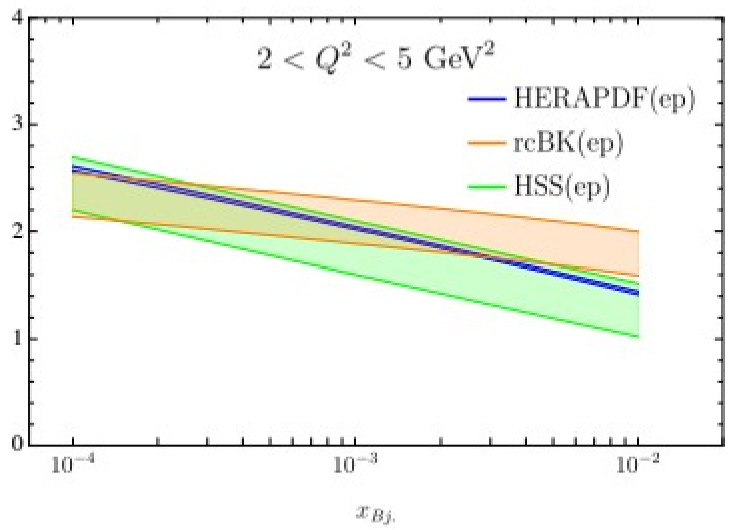
$$\partial_Y p_n(Y) = -\lambda n p_n(Y) + \lambda(n-1)p_{n-1}(Y)$$

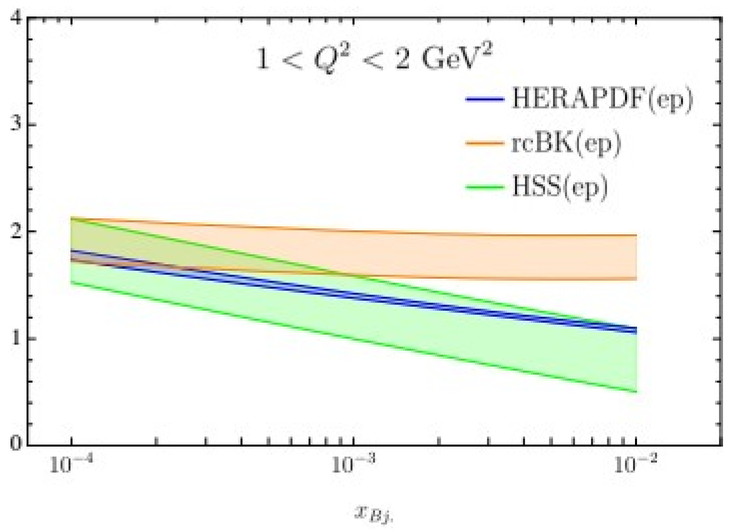
$$\partial_Y p_n(Y) = -\lambda n p_n(Y) + \lambda(n-1)p_{n-1}(Y)$$
$$+ \beta n(n+1)p_{n+1}(Y) - \beta n(n-1)p_n(Y)$$

Bondarenko, Motyka, Mueller, Shoshi, Xiao ‘07

Hagiwara,Hatta,Xiao,Yuan
Phys.Rev.D 97 (2018) 9, 094029

Small scales - prediction





Conclusions and outlook

- We show further evidences for the proposal for low x maximal entanglement entropy of proton constituents. See also
  M. Hentschinski, K.Kutak, R. Straka EPJC ’22,
  H. Hentschinski, D. Kharzeev. K. Kutak, Z. Tu PRL‘23

- The KL formalism after generalization to moving rapidity window describes data

- We related the fixed dipole size evolution equation to equation for probabilities that follow from Lanczos/Krylov construction

- We demonstrated that gluon density within this model corresponds to Krylov complexity

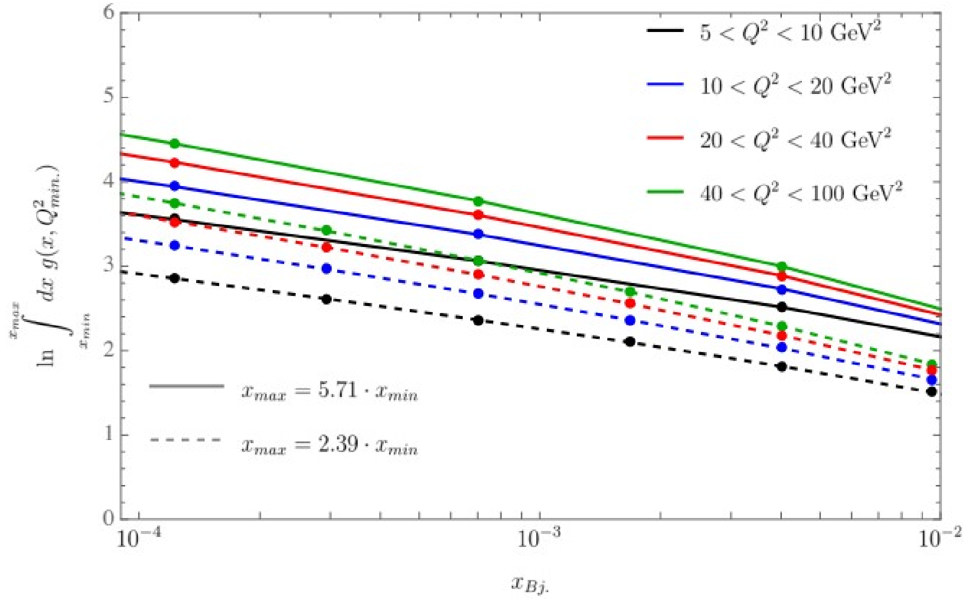
- We applied various quantum measures to the multiplicity densities of the dipole model. New handle to look for saturation.

$$[L_0, \phi(z)] = \left(z\frac{d}{dz} + h \right) \phi(z)$$ scaling

$$[L_{-1}, \phi(z)] = \frac{d}{dz}\phi(z)$$ translations

$$[L_1, \phi(z)] = \left(z^2\frac{d}{dz} + 2hz \right) \phi(z)$$ special conformal

Bining and KL formula



plot showing dependence of the result on the sie of bins if binig is naive

$$n_g(Q^2) = \int_0^1 dx g(x, Q^2)$$

Formal definition of number of gluons

$$n_g(\bar{x}) = \int_{x_{\min}}^{x_{\max}} dx g(x, Q^2)$$

$$\bar{x} \in [x_{\min}, x_{\max}]$$

$$\bar{x} = \frac{\int_{x_{\min}}^{x_{\max}} dx\, x g(x, Q^2)}{\int_{x_{\min}}^{x_{\max}} dx g(x, Q^2)}$$

average *x*

Data binning takes place in rapidity

$$\bar{n}_g(\bar{x}) = \frac{1}{y_{max} - y_{min}} \int_{y_{min}}^{y_{max}} dy \frac{dn_g}{dy} = \frac{n_g(y_{max}) - n_g(y_{min})}{y_{max} - y_{min}}$$

$$y_{max,min} = \ln 1/x_{min,max}$$

for small bins

$$\bar{n}_g(x, Q^2) = \frac{dn_g}{d\ln(1/x)} = xg(x, Q^2)$$

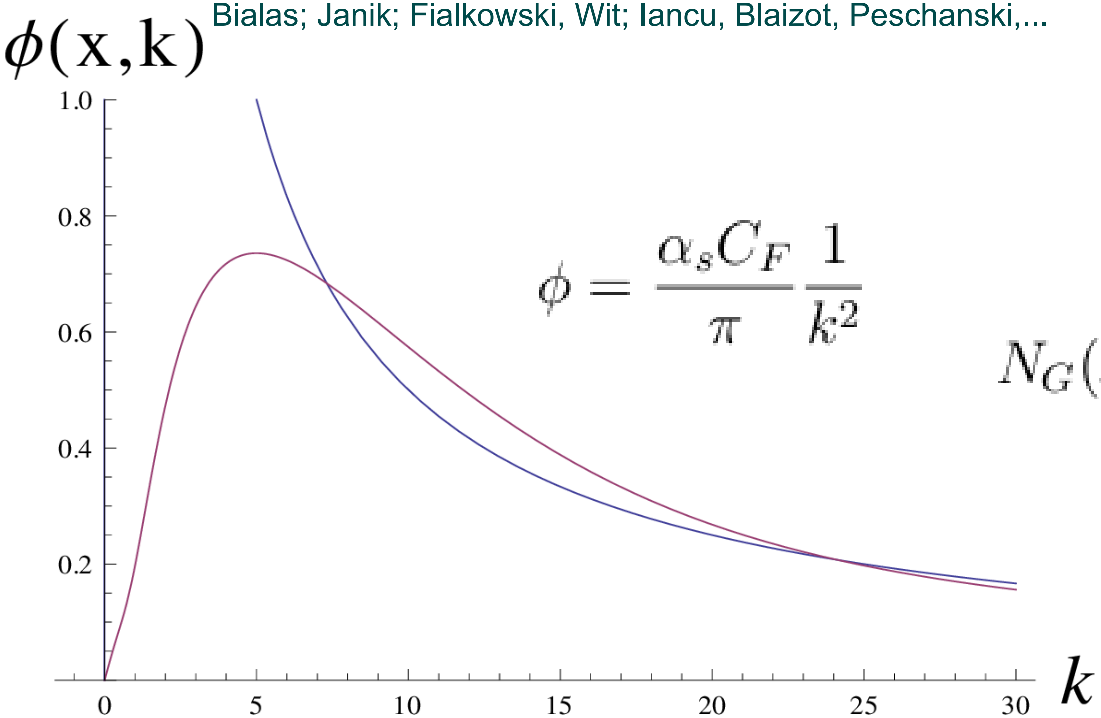
$$\langle \bar{n}(x, Q^2) \rangle_{Q^2} = \frac{1}{Q^2_{\max} - Q^2_{\min}} \int_{Q^2_{\min}}^{Q^2_{\max}} dQ^2 \left[xg(x, Q^2) + x\Sigma(x, Q^2) \right]$$

$$\langle S(x, Q^2) \rangle_{Q^2} = \ln \langle \bar{n}(x, Q^2) \rangle_{Q^2}$$

Gluon production and entropy – another assumptions

Kutak '11

Bialas; Janik; Fialkowski, Wit; Iancu, Blaizot, Peschanski,...



$$M_G(x) = Q_s(x)$$ energy dependent gluon's mass

$$M(x) = N_G(x) M_G(x)$$ mass of system of gluons

$$N_G(x) \equiv \frac{dN}{dy} = \frac{1}{S_\perp}\frac{d\sigma}{dy}$$ number of gluons

$$dE = TdS$$

$$dM = TdS$$

$$d\left[N_G(x)\, M_G(x)\right] = \frac{Q_s(x)}{2\pi} dS$$

Many-body interactions

Medium generated mass of gluon.
Framework of Hard Thermal Loops.

Entropy due to less dense hadron

$$S = \frac{6 C_F A_\perp}{\pi \alpha_s} Q_s^2(x) + S_0$$

$$S = 3\pi\left[N_G(x) + N_{G0}\right]$$

Similarly in QED. Cut on photon`s kt
Is equivalent to introducing mass.

In presented approach mass is not fixed it is x dependent