Exact Wigner function for chiral spirals

by Sudip Kumar Kar[1]
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Agenda

- Background
- Chiral spirals
- Solutions to chiral spirals
- **Wigner functions**
- **Twist in polarization**
- **Breakdown of semiclassical expansion**

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- Chiral spirals belong to a broad range of mean fields that are used to study pion condensation.[V. Schon and M. Thies, arXiv:hep-th/0008175.]
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  - Presented a self consistent solution that agreed with bulk properties of nuclear matter [B. Banerjee, N. Glendenning, and M. Gyulassy, Nuclear Physics A 361, 326 (1981).]
  - Investigation in various kinds of pion condensed phases in an effective chiral models [M. Kutschera, W. Broniowski, and A. Kotlorz, Nuclear Physics A 516, 566 (1990).]
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- Potential practical application was on Pulsars, although current considerations seem to rule out this possibility.
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Chiral symmetry and chiral models

- The symmetry between left and right handed fermions is called the chiral symmetry.
- One such example would be the NJL model with scalar and pseudoscalar condensates [S. P. Klevansky, Rev. Mod. Phys. 64, 649 (1992).]

$$\mathcal{L} = i\hbar\bar{\psi}\,\partial\!\!\!/\psi + G\Big[\underbrace{(\bar{\psi}\psi)^2}_{\text{scalar }(\hat{\sigma})} + \underbrace{(\bar{\psi}i\gamma_5\psi)^2}_{\text{pseudoscalar }(\hat{\pi})}\Big]. \tag{1}$$

($A\!\!\!/ = A_\mu\gamma^\mu$ and the γ matrices are taken here in the Dirac representation.)

- The $U_A(1)$ chiral transformation of fields is given by

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Chiral spirals

- We consider Dirac particles that interact with some externally given mean field for scalar and pseudoscalar condensates.
- Such systems are governed by the dynamical equation given below

$$\left[i\hbar\gamma_\mu\partial^\mu - \sigma - i\gamma_5\pi\right]\psi(x) = 0, \tag{2}$$

- In chiral spirals we assume a periodic form for the mean fields given as,

$$\sigma = M\cos(\phi), \qquad \pi = M\sin(\phi). \tag{3}$$

- Where, one may proceed with two kinds of forms for ϕ,

$$\frac{\boldsymbol{q}\cdot\boldsymbol{x}}{\hbar} \text{ where } q \text{ is momentum} \tag{4}$$

$$\boldsymbol{\kappa}\cdot\boldsymbol{x} \text{ where } \boldsymbol{\kappa} \text{ is a Wave vector} \tag{5}$$

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Why Spirals?

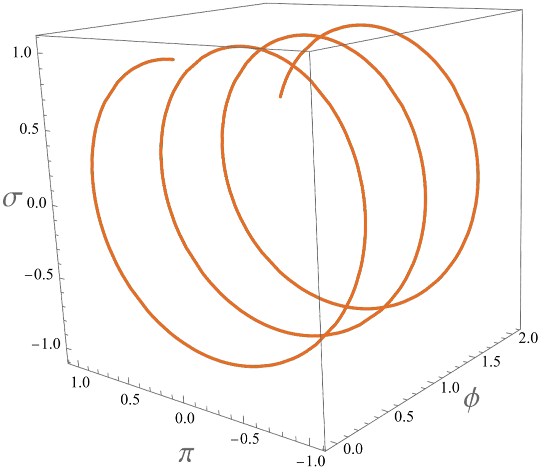

Figure: Spiral nature of the scalar and pseudoscalar condensate

Ansatz to solve chiral spirals

- The forms of the condensate turns our dynamical equation into a Dirac-like equation given as,

$$\left[i\hbar\gamma_\mu\partial^\mu - Me^{i\gamma_5(\boldsymbol{q}\cdot\boldsymbol{x})/\hbar}\right]\psi(x) = 0. \tag{6}$$

- Fortunately, the equation of motion can be solved exactly by the following ansatz, [F. Dautry and E. M. Nyman, Nuclear Physics A 319, 323 (1979).]

$$\psi_{\pm}(x) = \exp\left(-\frac{i\gamma_5}{2}\frac{\boldsymbol{q}\cdot\boldsymbol{x}}{\hbar}\right)\chi_{\pm}(\boldsymbol{p})\, e^{\mp ip\cdot x/\hbar}, \tag{7}$$

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Using the Pauli matrices, $\boldsymbol{\tau}$, we write

$$\overbrace{\begin{pmatrix} M - \frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{q} & \boldsymbol{\tau}\cdot\boldsymbol{p} \\ \boldsymbol{\tau}\cdot\boldsymbol{p} & -M - \frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{q} \end{pmatrix}}^{\text{“Positive” energies}}, \qquad \overbrace{\begin{pmatrix} -M + \frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{q} & \boldsymbol{\tau}\cdot\boldsymbol{p} \\ \boldsymbol{\tau}\cdot\boldsymbol{p} & M + \frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{q} \end{pmatrix}}^{\text{“Negative” Energies}}. \tag{8}$$

- We assume $\boldsymbol{q}$ to be along the z-axis and find the eigenvalues.
- The eigenvalues give the energy spectrum which features a split in energy for different spins ($q > 0$),

$$E_{\boldsymbol{p}}^{(r)} = \sqrt{\boldsymbol{p}^2 + q^2/4 + M^2 + (-1)^{r-1} q \sqrt{M^2 + (p^3)^2}} \qquad (r = 1, 2). \tag{9}$$

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Energy spectrum of the system

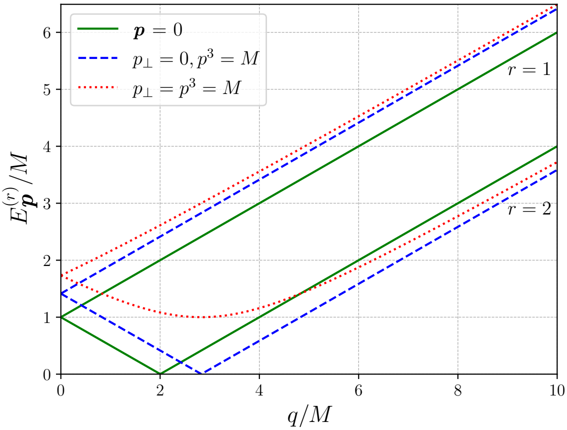

Figure: Energy to Mass ratio, $E_{\boldsymbol{p}}^{(r)}/M$ plotted as a function of q/M for two different values of spin

The split in the spectrum naturally creates polarization!

- One spinor is less energetic than the other, this could be a natural mechanism of spin polarization.
- To study spin polarization in a quantum context we require Wigner functions.
- The spinors may be used to compute the Wigner function with ease.
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The spinors

The positive and negative energy spinors are given as follows,

$$\chi_{\pm}^{(r)}(\boldsymbol{p}) = N_{\pm}^{(r)} \begin{pmatrix} \frac{E_{\boldsymbol{p}}^{(r)} \pm (-1)^r E_{\boldsymbol{p}}^{\parallel} \mp \frac{q}{2}}{p^1 + ip^2} \\ \pm \frac{p^3}{M + (-1)^r E_{\boldsymbol{p}}^{\parallel}} \\ \pm \frac{E_{\boldsymbol{p}}^{(r)} \pm (-1)^r E_{\boldsymbol{p}}^{\parallel} \mp \frac{q}{2}}{p^1 + ip^2} \frac{p^3}{M + (-1)^r E_{\boldsymbol{p}}^{\parallel}} \\ 1 \end{pmatrix}. \quad (10)$$

Where the factor $N_{\pm}^{(r)}$ has been added to normalize the spinors to,

$$\chi_{+}^{(r)^\dagger}(\boldsymbol{p})\chi_{+}^{(r)}(\boldsymbol{p}) = 2E_{\boldsymbol{p}}^{(r)}, \qquad \chi_{-}^{(r)^\dagger}(\boldsymbol{p})\chi_{-}^{(r)}(\boldsymbol{p}) = 2E_{\boldsymbol{p}}^{(r)}. \quad (11)$$

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Properties of the spinors

Orthogonality of the spinors is given as,

Chiral spiral	Free Dirac case
$\chi_+^{(r)^\dagger}(\boldsymbol{p})\,\chi_+^{(s)}(\boldsymbol{p}) = 2E_{\boldsymbol{p}}^{(r)}\delta^{rs}$	$u^{(r)^\dagger}(\boldsymbol{p})u^{(s)}(\boldsymbol{p}) = 2E_{\boldsymbol{p}}\delta^{rs}$
$\chi_-^{(r)^\dagger}(\boldsymbol{p})\,\chi_-^{(s)}(\boldsymbol{p}) = 2E_{\boldsymbol{p}}^{(r)}\delta^{rs}$	$v^{(r)^\dagger}(\boldsymbol{p})v^{(s)}(\boldsymbol{p}) = 2E_{\boldsymbol{p}}\delta^{rs}$
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$\bar{\chi}_+^{(r)}(\boldsymbol{p})\,\chi_-^{(s)}(\boldsymbol{p}) = 0$	$\bar{u}^{(r)}(\boldsymbol{p})v^{(s)}(\boldsymbol{p}) = 0$
$\bar{\chi}_-^{(r)}(\boldsymbol{p})\,\chi_+^{(s)}(\boldsymbol{p}) = 0$	$\bar{v}^{(r)}(\boldsymbol{p})u^{(s)}(\boldsymbol{p}) = 0$

While the completeness is given as (a and b are spinor indices),

$$\sum_{r=1}^{2}\frac{1}{2E_{\boldsymbol{p}}^{(r)}}\left[\chi_{+,a}^{(r)}(\boldsymbol{p})\chi_{+,b}^{(r)^\dagger}(\boldsymbol{p}) + \chi_{-,a}^{(r)}(-\boldsymbol{p})\chi_{-,b}^{(r)^\dagger}(-\boldsymbol{p})\right] = \delta_{ab}. \tag{12}$$

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While the completeness is given as (a and b are spinor indices),

$$\sum_{r=1}^{2} \frac{1}{2E_{\boldsymbol{p}}^{(r)}} \left[\chi_{+,\,a}^{(r)}(\boldsymbol{p})\chi_{+,\,b}^{(r)^\dagger}(\boldsymbol{p}) + \chi_{-,\,a}^{(r)}(-\boldsymbol{p})\chi_{-,\,b}^{(r)^\dagger}(-\boldsymbol{p})\right] = \delta_{ab}. \tag{12}$$

The spinor fields

The general field is thus given as,

$$\psi(x) = \sum_{r=1,2} \int \frac{d^3p}{(2\pi\hbar)^{3/2}} \frac{1}{\sqrt{2E_{\boldsymbol{p}}^{(r)}}} \left[u^{(r)}(\boldsymbol{p},\boldsymbol{x}) b_r(\boldsymbol{p}) e^{-\frac{i}{\hbar}p\cdot x} + v^{(r)}(\boldsymbol{p},\boldsymbol{x}) c_r^*(\boldsymbol{p}) e^{\frac{i}{\hbar}p\cdot x} \right], \tag{13}$$

with,

$$u^{(r)}(\boldsymbol{p},\boldsymbol{x}) = \exp\left(-\frac{i\gamma_5}{2}\frac{\boldsymbol{q}\cdot\boldsymbol{x}}{\hbar}\right)\chi_+^{(r)}(\boldsymbol{p}), \tag{14}$$

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Quantization of the spinor fields

- The field operator defined in previous slide satisfies the canonical equal-time (anti-)commutation relations,

$$\{\psi_a(t,\boldsymbol{x}), \psi_b^\dagger(t,\boldsymbol{y})\} = \delta_{ab}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \tag{16}$$

$$\{\psi_a(t,\boldsymbol{x}), \psi_b(t,\boldsymbol{y})\} = \{\psi_a^\dagger(t,\boldsymbol{x}), \psi_b^\dagger(t,\boldsymbol{y})\} = 0, \tag{17}$$

- Provided that the creation and annihilation operators satisfy the (anti-)commutation relations

$$\left\{b_r(\boldsymbol{p}), b_s^\dagger(\boldsymbol{p}')\right\} = \left\{c_r(\boldsymbol{p}), c_s^\dagger(\boldsymbol{p}')\right\} = \delta^{(3)}\left(\boldsymbol{p}-\boldsymbol{p}'\right)\delta_{rs}, \tag{18}$$

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The $\boldsymbol{p} \to 0$ limit

The $\boldsymbol{p} \to 0$ limit reveals the relationship between the index (r) and the direction of spin polarization.

With $\boldsymbol{p} = (p^1, p^2, p^3) = (|\boldsymbol{p}| \sin\theta \cos\phi, |\boldsymbol{p}| \sin\theta \sin\phi, |\boldsymbol{p}| \cos\theta)$, where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ are the polar and azimuthal angles, respectively, One may take the $|\boldsymbol{p}| \to 0$ limit, to get the following spinors,

$$\chi_+^{(1)}(\boldsymbol{p} \to 0) = -\,\mathrm{sgn}(\cos\theta)\sqrt{2M+q}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{“Down”} \tag{20}$$

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The $\boldsymbol{q} \to 0$ (homogeneous) limit

The $\boldsymbol{q} \to 0$ limit is important for comparison with other expressions found in the literature. In this limit, the eigenvalues reduce to,

$$E(\boldsymbol{p}) = \sqrt{M^2 + \boldsymbol{p}^2} \tag{24}$$

The $q \to 0$ limit of eigenvalues is straightforward.
The eigenvectors become,

$$\chi_{\pm}^{(r)}(\boldsymbol{p}, q \to 0) = \sqrt{\frac{(p^1)^2 + (p^2)^2}{2E_{\boldsymbol{p}}^{||}}} \begin{pmatrix} \frac{\sqrt{E(\boldsymbol{p}) \pm (-1)^r E_{\boldsymbol{p}}^{||}} \sqrt{E_{\boldsymbol{p}}^{||} + (-1)^r M}}{p^1 + ip^2} \\ \pm(-1)^r \frac{p^3}{\sqrt{E(\boldsymbol{p}) \pm (-1)^r E_{\boldsymbol{p}}^{||}} \sqrt{E_{\boldsymbol{p}}^{||} + (-1)^r M}} \\ \pm(-1)^r \frac{p^3 \sqrt{E(\boldsymbol{p}) \pm (-1)^r E_{\boldsymbol{p}}^{||}}}{(p^1 + ip^2) \sqrt{E_{\boldsymbol{p}}^{||} + (-1)^r M}} \\ \frac{\sqrt{E_{\boldsymbol{p}}^{||} + (-1)^r M}}{\sqrt{E(\boldsymbol{p}) \pm (-1)^r E_{\boldsymbol{p}}^{||}}} \end{pmatrix} \tag{25}$$

This does not match with the traditional free Dirac case!

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The $\boldsymbol{q} \to 0$ (homogeneous) limit

- The free Dirac limit is realized when a linear combination of these spinors is taken.
- Using the matrices U and V we represent these linear combinations,

$$\chi_{+}^{(r)}(\boldsymbol{p}, \boldsymbol{q} = 0) = U^{rr'} u_d^{r'}(\boldsymbol{p}) \qquad \chi_{-}^{(r)}(\boldsymbol{p}, \boldsymbol{q} = 0) = V^{rr'} v_d^{r'}(\boldsymbol{p}) \tag{26}$$

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Wigner functions

The Wigner functions are quasi-probablility distribution functions that are used to describe phase space of quantum mechanical systems. For spin-1/2 systems this looks like,

$$W_{ab}(x,k) = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{ik\cdot y}{\hbar}} \left\langle \bar{\psi}_b\left(x+\frac{y}{2}\right)\psi_a\left(x-\frac{y}{2}\right)\right\rangle . \tag{27}$$

The non-zero expectation values are,

$$\left\langle b_s^\dagger(\boldsymbol{p}')b_r(\boldsymbol{p})\right\rangle = \delta_{sr}\delta^{(3)}(\boldsymbol{p}'-\boldsymbol{p})f(E_{\boldsymbol{p}}^{(r)}-\mu_r), \tag{28}$$

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$$f(E_{\boldsymbol{p}}^{(r)}-\mu_r) = f_{\boldsymbol{p}}^{(r)}, \qquad f(E_{\boldsymbol{p}}^{(r)}+\bar{\mu}_r) = \bar{f}_{\boldsymbol{p}}^{(r)}, \tag{30}$$

Where, $\mu_r = \mu_B + (-1)^{r-1}\mu_{\mathfrak{s}}$ for particles
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It also useful to rewrite, Eqs. (28-29) in a matrix notation,

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Clifford algebra decomposition of the Wigner function

The gamma matrices satisfy the Clifford algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} 1_{4\times 4} \tag{35}$$

This allows one to decompose any 4×4 matrix into to a linear combination of the matrices, $\Gamma = \{1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \Sigma_{\mu\nu}\}$, where $\Sigma_{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$
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[D. Vasak, M. Gyulassy, and H. T. Elze, Annals Phys. 173, 462 (1987)],

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Each component in this decomposition can be obtained by taking the trace of the product of the Wigner function and the corresponding element of the basis set Γ.

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Clifford algebra decomposition of the Wigner function

The gamma matrices satisfy the Clifford algebra,

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu} 1_{4\times 4} \tag{35}$$

This allows one to decompose any 4×4 matrix into to a linear combination of the matrices, $\Gamma = \{1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \Sigma_{\mu\nu}\}$, where $\Sigma_{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]$
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Vector and Pseudovector Currents

Computation of these currents reveal that the space-like part of the Vector currents vanish leaving only the time-like component non-zero,

$$V^0(x) = \int d^4k\, \mathcal{V}^0(x,k) = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi\hbar)^3} \left(f_{\boldsymbol{k}}^{(r)} - \bar{f}_{\boldsymbol{k}}^{(r)} \right). \tag{37}$$

Similarly, for the axial part, the time like component and two of the space-like component vanish,

$$A^3(x) = \int d^4k\, \mathcal{A}^3(x,k) = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi\hbar)^3} \frac{(-1)^r E_{\boldsymbol{k}}^{\parallel}}{E_{\boldsymbol{k}}^{(r)}} \left[1 + \frac{(-1)^{r-1} q}{2E_{\boldsymbol{k}}^{\parallel}} \right] \left(f_{\boldsymbol{k}}^{(r)} + \bar{f}_{\boldsymbol{k}}^{(r)} \right). \tag{38}$$

The expression above agrees with the formula obtained in [M. Kutschera, W. Broniowski, and A. Kotlorz, Nuclear Physics A 516, 566 (1990).] up to the internal degrees of freedom connected with flavor and color.

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Variation of $A^3(x)$ with inhomogeneity factor q

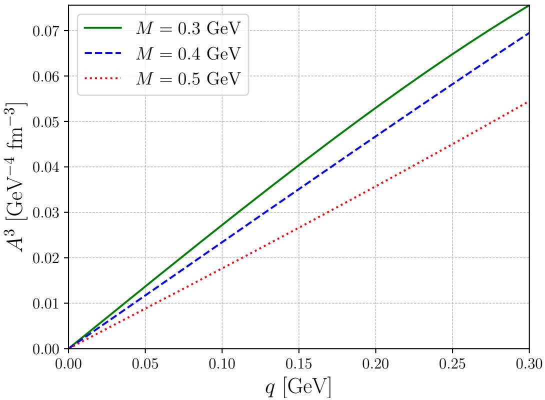

Figure: Third component of the axial vector as a function of the inhomogeneity factor q for different values of effective masses at constant chemical potential and temperature of $\mu = 0.1$ GeV and $T = 0.15$ GeV, respectively.

Variation of $A^3(x)$ with baryon chemical potential

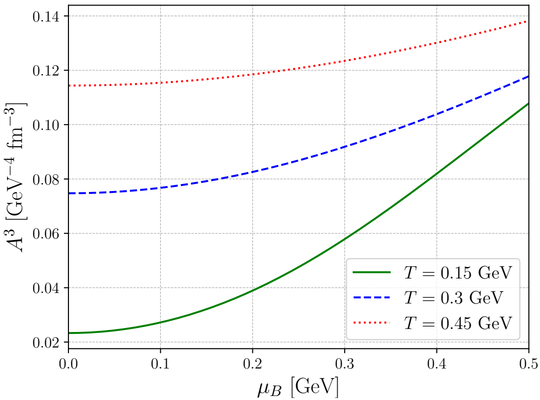

Figure: Third component of the axial vector as a function of the chemical potential μ for different values of temperature at constant mass and inhomogeneity factor of $M = 0.3$ GeV and $q = 0.1$ GeV, respectively.

Twist of spin polarization in $q \to 0$ (homogeneous) limit

The scalar, pseudoscalar and vector component agree with free Dirac case in the $q \to 0$ limit

The axial vector component in the $q \to 0$ disagrees with free Dirac case.

This situation is identical to the one faced when taking the degenerate limit of a non degenerate theory.

Since in the $q \to 0$ limit the eigenvectors are actually linear combination of the free Dirac spinors, the net effect is to twist the spin polarization direction.

Expected result in the free Dirac case

$$\zeta_{\boldsymbol{p}} = (0, 0, 1) \tag{39}$$

Twisted polarization

$$\zeta_{\boldsymbol{p}} = -\frac{1}{E_p^{\parallel}(E_{\boldsymbol{p}} + M)} \left(p^1 p^3,\, p^2 p^3,\, E_p^{\parallel 2} + E_{\boldsymbol{p}} M \right) \tag{40}$$

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Semiclassical expansion of the Wigner function

In the case of the periodic chiral condensate considered in this work, the semiclassical expansion leads to the equation

[W. Florkowski, J. Hufner, S. Klevansky, and L. Neise, Annals of Physics 245, 445–463 (1996).]

$$\left[\left(k^\mu + \frac{i\hbar}{2}\partial^\mu\right)\gamma_\mu - \sigma(x) + \frac{i\hbar}{2}\partial_\mu\sigma(x)\partial_k^\mu \right.$$
$$\left. - i\gamma_5\pi(x) - \frac{\hbar}{2}\gamma_5\partial_\mu\pi(x)\partial_k^\mu\right] W(x,k) = 0. \qquad (41)$$

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the decomposition of the Wigner function defined by Eq. (36), leads to a system of coupled equations of the form:

$$K^\mu \mathcal{V}_\mu - \sigma\mathcal{F} + \pi\mathcal{P} = \frac{i\hbar}{2}\Big[(\partial_\mu \pi)(\partial_k^\mu \mathcal{P}) - (\partial_\mu\sigma)(\partial_k^\mu \mathcal{F})\Big], \tag{42}$$

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$$iK^\mu \mathcal{P} - K_\nu \tilde{\mathcal{S}}^{\nu\mu} - \sigma\mathcal{A}^\mu + i\pi\mathcal{V}^\mu = \frac{i\hbar}{2}\Big[i(\partial_\nu \pi)(\partial_k^\nu \mathcal{V}^\mu) - (\partial_\nu\sigma)(\partial_k^\nu \mathcal{A}^\mu)\Big], \tag{45}$$

$$i(K^\mu\mathcal{V}^\nu - K^\nu\mathcal{V}^\mu) - \epsilon^{\mu\nu\tau\sigma}K_\tau\mathcal{A}_\sigma - \pi\tilde{\mathcal{S}}^{\mu\nu} + \sigma\mathcal{S}^{\mu\nu} = \frac{i\hbar}{2}\Big[(\partial_\gamma\sigma)(\partial_k^\gamma \mathcal{S}^{\mu\nu}) - (\partial_\gamma\pi)\left(\partial_k^\gamma \tilde{\mathcal{S}}^{\mu\nu}\right)\Big]. \tag{46}$$

Here, $K^\mu = k^\mu + \frac{i\hbar}{2}\partial^\mu$ and $\tilde{\mathcal{S}}$ is the dual tensor to the tensor $\mathcal{S}$, namely

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Semiclassical expansion of the Wigner function

within the semiclassical expansion, one solves these equations by expanding all the coefficients of the Wigner function as a series in $\hbar$,

$$C = C_{(0)} + \hbar C_{(1)} + \hbar^2 C_{(2)} + \dots \quad , \tag{48}$$

where C is one of the coefficients from the set $\{\mathcal{F}, \mathcal{P}, \mathcal{V}^\mu, \mathcal{A}^\mu, \mathcal{S}^{\mu\nu}\}$.

This when applied to the first dynamical equation yields,

$$k_\mu \mathcal{V}^\mu_{(0)} - \sigma_{(0)} \mathcal{F}_{(0)} + \pi_{(0)} \mathcal{P}_{(0)} = 0 \tag{49}$$

upto first order in $\hbar$.

A quick check of the case with $k = (k^0, 0)$ and $q \neq 0$ shows that the equation is not satisfied.

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- The canonical quantization of these fields was verified, thus ensuring their applicability in quantum cases.
- The Wigner function and all its components were computed exactly.
- The vector and axial current were extracted from these components and then used to compute currents.
- The $q \to 0$ limit of the vector and axial vector currents were analyzed, the axial vector current, in particular, displayed a curious twist from the expected free case.
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Supplemental slides

Normalisation for the spinors

$$N_{\pm}^{(r)} = \sqrt{2E_{\boldsymbol{p}}^{(r)}}\left[1+\frac{\left(E_{\boldsymbol{p}}^{(r)} \pm (-1)^{r} E_{\boldsymbol{p}}^{\parallel} \mp \frac{q}{2}\right)^{2}}{(p^{1})^{2}+(p^{2})^{2}}\right]^{-1/2}\left[1+\frac{(p^{3})^{2}}{\left(M+(-1)^{r} E_{\boldsymbol{p}}^{\parallel}\right)^{2}}\right]^{-1/2} . \tag{50}$$

Non-degenerate to degenerate reduction

To see why this is not something unusual, consider the following example,

$$\begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} \text{ has eigenvectors given by } |1 \pm \epsilon\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \tag{51}$$

The expectation values, $\langle 1+\epsilon|\tau^3|1+\epsilon\rangle$ and $\langle 1-\epsilon|\tau^3|1-\epsilon\rangle$ are zero (Here, $\tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$).

which in the $\epsilon \to 0$ limit becomes,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ which has eigenvectors given by } |1\rangle_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{52}$$

and compute the same expectation value as the previous case to obtain,

$$_1\langle 1|\tau^3|1\rangle_1 = 1 \qquad _2\langle 1|\tau^3|1\rangle_2 = -1 \tag{53}$$

Non-degenerate to degenerate reduction

To see why this is not something unusual, consider the following example,

$$\begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} \text{ has eigenvectors given by } |1 \pm \epsilon\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \tag{51}$$

The expectation values, $\langle 1 + \epsilon | \tau^3 | 1 + \epsilon \rangle$ and $\langle 1 - \epsilon | \tau^3 | 1 - \epsilon \rangle$ are zero (Here, $\tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$).
which in the $\epsilon \to 0$ limit becomes,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ which has eigenvectors given by } |1\rangle_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{52}$$

and compute the same expectation value as the previous case to obtain,

$$_1\langle 1 | \tau^3 | 1 \rangle_1 = 1 \qquad _2\langle 1 | \tau^3 | 1 \rangle_2 = -1 \tag{53}$$