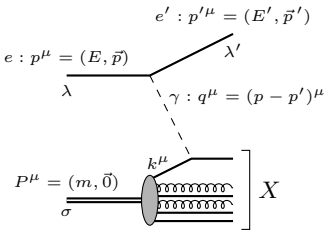
Review on Color Glass Condensate (CGC) results.

F. Cougoulic (Jagiellonian University)

AGH - November 14, 2025.

"The Diagram":



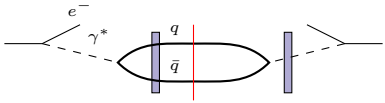
Kinematic invariants

$$Q^2 \equiv -q^2, \qquad (\gamma\text{-virtuality})$$

$$x_B \equiv \frac{Q^2}{2P \cdot q} = \frac{Q^2}{\hat{s} + Q^2 - m^2} \qquad (\text{Bjorken-x}),$$

$$y \equiv \frac{E - E'}{E}, \qquad (\text{Inelasticity})$$

Factorization in the CGC framework: study the small-x behavior $\rightarrow$ High Energy limit.



$$\sigma_{T,L}^{\gamma^* p}(x, Q^2) = \sigma_0 \int d^2\underline{r} \int_0^1 dz \, |\Psi_{T,L}(\underline{r}, z, Q^2) \left[1 - S_{\underline{r}}^{(\eta = \ln 1/x)}\right] \tag{2}$$

Taking the HE limit: consider the scattering of a probe on some (localized) external field,

- S-matrix $S_{\beta\alpha} = \langle \beta_{out} | \alpha_{in} \rangle$ (finite energy).
- Evolution operator U write $S_{\beta\alpha} = \langle \beta_{in} | U(\infty, -\infty) | \alpha_{in} \rangle = \langle \beta_{in} | T\, e^{i \int d^4x \mathcal{L}_{int}(x)} | \alpha_{in} \rangle$.
- Apply a boost in the longitudinal direction to the states α, β: $|\alpha_{in}\rangle \to e^{-i\,w K^3} |\alpha_{in}\rangle$, and extract the limit of infinite boost of the amplitude.

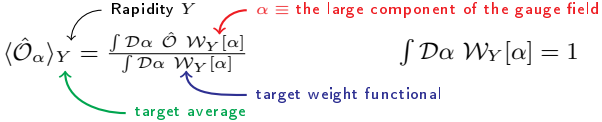
Eikonal scattering: as a consequence to the infinite boost limit,

- Support of the external field is contracted: $L \to 0$; and its configuration is frozen ("glass"): $\tau_{int} \ll \tau_{probe}$.
- Component of the external gauge field are ordered in eikonality: $A^{\pm} \gg A_{\perp} \gg A^{\mp}$
- Wave function is evolved from $-\infty$ to 0, then the scattering is a phase rotation encoded into Wilson lines.

Beyond Eikonal, **Sub-Eikonal**, **next-to-eikonal**, ... : consists of relaxing the above limits.

Where is the dynamics?

Expectation of an operator made of Wilson Lines (WL) at a given rapidity



Evolution $Y \to Y + \delta Y$

$$\partial_Y W_Y[\alpha] = H_{JIMWLK} \cdot W_Y[\alpha] \tag{3}$$

Evolution Hamiltonian

$$H_{JIMWLK} \equiv \frac{\alpha_s}{2\pi^2} \int \mathrm{d}^2\underline{x}\, \mathrm{d}^2\underline{y}\, \mathrm{d}^2\underline{z}\ \frac{(\underline{x}-\underline{z})\cdot(\underline{y}-\underline{z})}{(\underline{x}-\underline{z})^2(\underline{y}-\underline{z})^2} \left(2U_{\underline{z}} - U_{\underline{x}} - U_{\underline{y}}\right)^{ba} (ig)^{-2} \frac{\delta}{\delta\alpha^a(x^- < 0, \underline{x})} \frac{\delta}{\delta\alpha^b(y^- > 0, \underline{y})} \tag{4}$$

$\Rightarrow$ Acting on the dipole operator $S = \mathrm{tr}\, VV^\dagger$ and taking the large N_c limits yields BK equation.

Initial condition for the evolution:

- GBW: the dipole reads $S^{GBW}(r_\perp) = \exp(-r_\perp^2 Q_s^2)$
- MV: the dipole reads $S^{MV}(r_\perp) = \exp(-r_\perp^2 Q_s^2 \ln \frac{1}{r\Lambda})$

An overview of recent developments

Accuracy of CGC calculations

- Next-to-leading order
- Beyond Eikonal approximation
- HE evolution and Collinear evolution

Phenomenology using the CGC

- Many dijets
- Meson production

Disclaimer: This is not an exhaustive review.

Part 1: Improving the accuracy of CGC calculations

Next-to-leading order: JIMWLK kernels

A. Kovner, M. Lublinsky, Y. Mulian - JHEP 08 (2014) 114 / JHEP 05 (2017) 097

Evolution of an operator $\hat{\mathcal{O}}$ over a rapidity interval δY:

$$\hat{\mathcal{O}} \longrightarrow e^{-\delta Y H_{JIMWLK}} \hat{\mathcal{O}}, \qquad \text{where } H_{JIMWLK} = H^{LO}_{JIMWLK}(\alpha_s) + H^{NLO}_{JIMWLK}(\alpha_s^2) + \cdots \quad (5)$$

$$H^{LO}_{JIMWLK} = \frac{\alpha_s}{2\pi^2} \int_{\mathbf{x},\,\mathbf{y},\,\mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} \left[J^a_L(\mathbf{x})\, J^a_L(\mathbf{y}) \;+\; J^a_R(\mathbf{x})\, J^a_R(\mathbf{y}) \;-\; 2 J^a_L(\mathbf{x})\, S^{ab}_A(\mathbf{z})\, J^b_R(\mathbf{y}) \right]. \quad (6)$$

$$\begin{aligned}
H^{NLO}_{JIMWLK} &= \int_{\mathbf{x},\mathbf{y},\mathbf{z}} K_{JSJ}(\mathbf{x},\mathbf{y},\mathbf{z}) \left[J^a_L(\mathbf{x}) J^a_L(\mathbf{y}) + J^a_R(\mathbf{x}) J^a_R(\mathbf{y}) - 2 J^a_L(\mathbf{x}) S^{ab}_A(\mathbf{z}) J^b_R(\mathbf{y}) \right] \\
&+ \int_{\mathbf{x},\,\mathbf{y},\mathbf{z},\mathbf{z}'} K_{JSSJ}(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') \left[f^{abc} f^{def} J^a_L(\mathbf{x}) S^{be}_A(\mathbf{z}) S^{cf}_A(\mathbf{z}') J^d_R(\mathbf{y}) \;- N_c J^a_L(\mathbf{x}) S^{ab}_A(\mathbf{z}) J^b_R(\mathbf{y}) \right] \\
&+ \int_{\mathbf{x},\,\mathbf{y},\,\mathbf{z},\,\mathbf{z}'} K_{q\bar{q}}(\mathbf{x},\,\mathbf{y},\,\mathbf{z},\,\mathbf{z}') \left[2\, J^a_L(\mathbf{x})\, tr[S^\dagger(\mathbf{z})\, t^a\, S(\mathbf{z}')\, t^b]\, J^b_R(\mathbf{y}) \;-\; J^a_L(\mathbf{x})\, S^{ab}_A(\mathbf{z})\, J^b_R(\mathbf{y}) \right] \\
&+ \int_{\mathbf{w},\mathbf{x},\,\mathbf{y},\,\mathbf{z},\,\mathbf{z}'} K_{JJSSJ}(\mathbf{w},\mathbf{x},\,\mathbf{y},\,\mathbf{z},\,\mathbf{z}')\, f^{acb} \left[J^d_L(\mathbf{x})\, J^e_L(\mathbf{y})\, S^{dc}_A(\mathbf{z})\, S^{eb}_A(\mathbf{z}')\, J^a_R(\mathbf{w}) \right. \\
&\left. - J^a_L(\mathbf{w})\, S^{cd}_A(\mathbf{z})\, S^{be}_A(\mathbf{z}')\, J^d_R(\mathbf{x})\, J^e_R(\mathbf{y}) + \frac{1}{3}(J^c_L(\mathbf{x})\, J^b_L(\mathbf{y})\, J^a_L(\mathbf{w}) \;-\; J^c_R(\mathbf{x})\, J^b_R(\mathbf{y})\, J^a_R(\mathbf{w})) \right] \\
&+ \int_{\mathbf{w},\mathbf{x},\,\mathbf{y},\,\mathbf{z}} K_{JJSJ}(\mathbf{w},\mathbf{x},\,\mathbf{y},\,\mathbf{z})\, f^{bde} \left[J^d_L(\mathbf{x})\, J^e_L(\mathbf{y})\, S^{ba}_A(\mathbf{z})\, J^a_R(\mathbf{w}) \right. \\
&\left. - J^a_L(\mathbf{w})\, S^{ab}_A(\mathbf{z})\, J^d_R(\mathbf{x})\, J^e_R(\mathbf{y}) + \frac{1}{3}(J^d_L(\mathbf{x})\, J^e_L(\mathbf{y})\, J^b_L(\mathbf{w}) \;-\; J^d_R(\mathbf{x})\, J^e_R(\mathbf{y})\, J^b_R(\mathbf{w})) \right].
\end{aligned} \quad (7)$$

- NLO Hamiltonian has rich Lorentz and color structure: RG / DGLAP / ...
- Still lacking a numerical implementation.

Status of the numerical implementations

Forefront (unpolarized - eikonal)

@eik	BK	JIMWLK
LO	✓	✓
r.c.	✓	✓
k.c.	✓	✗
NLO	✓/✗	✗
N^2LO	✗	✗

Personal bias: (polarized)

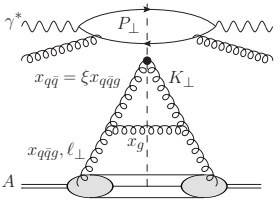
@sub-eik	Large-N_C	Large-$N_c\&N_f$	All-N_c
LO	✓	✓	✗
r.c.	✗	✗	✗
NLO	✗	✗	✗

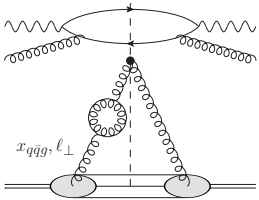
There is still a lot of fun to have!

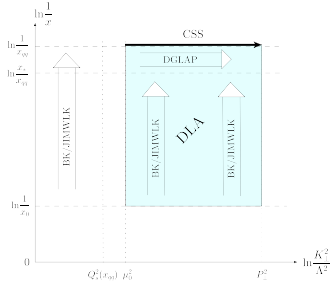
Next-to-leading order: Evolution of gluon TMD at small-x

P. Caucal and E. Iancu - Phys. Rev. D 111 (2025) 7, 074008

- Compute NLO correction to the production of pair of hard jets in DIS.
- NLO corrections include the JIMWLK evolution but also the DGLAP/CSS evolution.
- Expect all three type of evolution to be relevant for $P_\perp \gg K_\perp \gg Q_s(x)$.
  Propose a framework to encompase all three evolution.
- Confirm: Sudakov effect can be ressumed according to CSS-eqs for the TMD.
- The dijet relative momentum $P_\perp$ sets the scales for both CSS-eqs (specific trajectory in CSS $\mu - \xi$ plane).







Next-to-leading order: with/without Sudakov

P. Caucal, F. Salazar, B. Schenke, T. Stebel, R. Venugopalan - Phys. Rev. Lett. 132 (2024) 8, 081902

- Back-to-back dijet in DIS at NLO within the CGC framework
- Result expressed as:

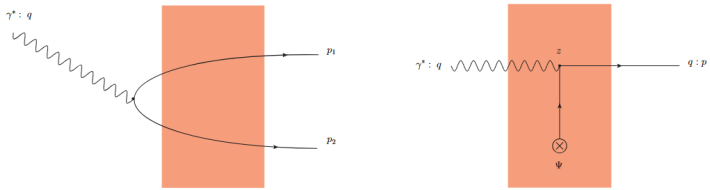
$$\left\langle \mathrm{d}\sigma_{\mathrm{LO}}^{(0),\lambda,\mathrm{b2b}} + \alpha_s \mathrm{d}\sigma_{\mathrm{NLO}}^{(0),\lambda,\mathrm{b2b}} \right\rangle_{\eta_f} = \mathcal{H}_{\mathrm{LO}}^{0,\lambda} \int \frac{\mathrm{d}^2\boldsymbol{B}_\perp}{(2\pi)^2} \int \frac{\mathrm{d}^2\boldsymbol{r}_{bb'}}{(2\pi)^2} e^{-i\boldsymbol{q}_\perp\cdot\boldsymbol{r}_{bb'}} \hat{G}^0_{\eta_f}(\boldsymbol{r}_{bb'},\mu_0) \Bigg\{ 1 + \frac{\alpha_s(\mu_R)}{\pi} \Big[\underbrace{-\frac{N_c}{4} \ln^2\left(\frac{\boldsymbol{P}_\perp^2 \boldsymbol{r}_{bb'}^2}{c_0^2}\right)}_{\text{Sudakov double log}}$$

$$\underbrace{-s_L \ln\left(\frac{\boldsymbol{P}_\perp^2 \boldsymbol{r}_{bb'}^2}{c_0^2}\right) + \beta_0 \ln\left(\frac{\mu_R^2 \boldsymbol{r}_{bb'}^2}{c_0^2}\right)}_{\text{Sudakov single logs}} + \frac{N_c}{2} f_1^\lambda(\chi,z_1,R,\eta_f) + \frac{1}{2N_c} f_2^\lambda(\chi,z_1,R) \Big] \Bigg\}$$

$$+ \frac{\alpha_s(\mu_R)}{\pi} \mathcal{H}_{\mathrm{LO}}^{0,\lambda} \int \frac{\mathrm{d}^2\boldsymbol{B}_\perp}{(2\pi)^2} \int \frac{\mathrm{d}^2\boldsymbol{r}_{bb'}}{(2\pi)^2} e^{-i\boldsymbol{q}_\perp\cdot\boldsymbol{r}_{bb'}} \hat{h}^0_{\eta_f}(\boldsymbol{r}_{bb'},\mu_0) \left\{ \frac{N_c}{2} \left[1 + \ln(R^2)\right] - \frac{1}{2N_c} \ln(z_1 z_2 R^2) \right\} . \quad (11)$$

- Up to correction of order $q_\perp^2/P_\perp^2$, $Q_s^2/P_\perp^2$.
- $\hat{G}^0$ - unpol WW gluon TMD / $\hat{h}$ - linearly pol WW gluon TMD
- Correct Sudakov implies the introduction of lifetime ordering for the action of the JIMWLK kernel on the WW TMD.

- Compute the SIDIS cross-section

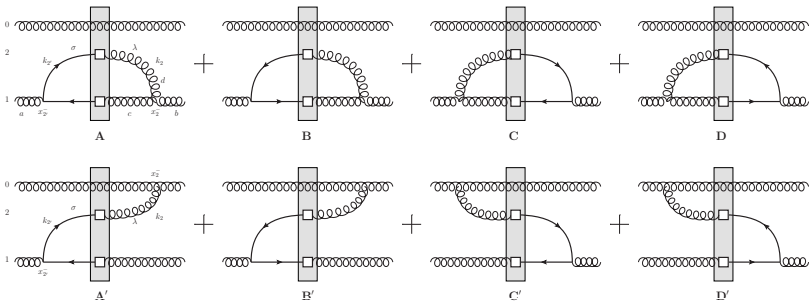


- Beyond eikonal approximation in the CGC. Inclusion of the quark background field.
- Directly relates to the (valence) quark TMD:
  $\propto \langle P_{tar}|\overline{\Psi}(z^+, b_\perp)\gamma^- V^\dagger[\infty, z^+; b_\perp]V[\infty, 0; 0_\perp]\Psi(0^+, 0_\perp)|P_{tar}\rangle$
- To be contrasted to the eikonal contribution where one recovers the sea quark TMD.

Beyond Eikonal approximation: Spin physics

J. Borden, Y. Kovchegov, M. Li - JHEP 09 (2024) 037

- Helicity evolution at small-x involves sub-eikonal operators.
- Evolution equations are double logarithmic $\int \frac{dz}{z} \int \frac{\mathrm{d}^2 \underline{w}}{(\underline{w}-\underline{x}_i)^2}$
- Modifies previous results [JHEP 07 (2022) 095] in the Veneziano large $N_c \& N_f$ limit.
- Involves fermion-boson $\langle \overline{\psi(y) a^\mu(x)} \rangle$ propagators in a background / Transition operators.
- Relevant for flavor singlet evolution and flavor non-singlet.



- Agreement with existing 3-loop splitting functions for $\Delta P_{qq}, \Delta P_{gg}$ in $\overline{MS}$ and (BER IRE).
- ΔP_{qg} and ΔP_{gq} differs at $\mathcal{O}(\alpha_s)$ with $\overline{MS}$. Attributed to scheme difference.

Prediction for the polarized splitting function at 4-loops at small-x.

$$\Delta P_{qq}^{(3)}(x) = \left(\frac{\alpha_s N_c}{4\pi}\right)^4 \frac{1}{720}\left(5 - 748\frac{N_f}{N_c} + 80\frac{N_f^2}{N_c^2}\right)\ln^6\frac{1}{x}, \tag{8a}$$

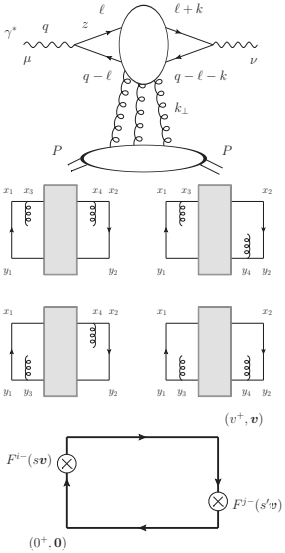
$$\Delta P_{qG}^{(3)}(x) = -\left(\frac{\alpha_s N_c}{4\pi}\right)^4 \frac{1}{360}\frac{N_f}{N_c}\left(1213 - 224\frac{N_f}{N_c}\right)\ln^6\frac{1}{x}, \tag{8b}$$

$$\Delta P_{Gq}^{(3)}(x) = \left(\frac{\alpha_s N_c}{4\pi}\right)^4 \frac{1}{360}\left(1213 - 224\frac{N_f}{N_c}\right)\ln^6\frac{1}{x}, \tag{8c}$$

$$\Delta P_{GG}^{(3)}(x) = \left(\frac{\alpha_s N_c}{4\pi}\right)^4 \frac{1}{180}\left(1984 - 549\frac{N_f}{N_c} + 20\frac{N_f^2}{N_c^2}\right)\ln^6\frac{1}{x}. \tag{8d}$$

HE evolution and Collinear evolution: finite length operators

R. Boussarie and Y. Mehtar-Tani - JHEP 07 (2022) 080



- Consider DIS cross section
- Aim: factorization valid in both the Bjorken limit ($Q^2 \to \infty$ while x_{Bj} finite) and the Regge limit ($s \to \infty$ equiv. $x_{Bj} \to 0$ while Q^2 finite)
- Consider the hadronic tensor $W^{\mu\nu}$ where the the propagators are in the bg of the *eikonal* gauge field $A^-(x^2, x_\perp)$.
- Beyond the usual approximation $\tau_{q\bar{q}} \gg \tau_{int}$
- Medium length L^+ identified by the very first and very last interaction
- $A^-(x^+, y_\perp) - A^-(x^-, x_\perp) \to \int_{x_\perp}^{y_\perp} dw^i F^{i-}(x^+, w_\perp)$
- Finite length operators: box with two field strength tensors insertion on transverse links.
- Limiting cases: Bjorken limit $v_\perp \to 0_\perp$ gives collinear pdf, Regge limit gives dipole cross section.

- Starting from the NLO JIMWLK Hamiltonian, study the the transverse logs associated to β_0.
- *"Not all that is β_0 is β-function"*
- $2\beta_0$ contribution in front of $\ln \mu^2_{\overline{MS}}$. Half of it is associated to DGLAP-like evolution (low-x pole is subtracted), as opposed to the running coupling.
- Implemented into JIMWLK hamiltonian by introducing *dressed* Wilson lines $\mathbb{S}^{ab}_Q$ to order α_s with resolution Q:

$$\mathbb{S}^{ab}_Q(\mathbf{z}) = S^{ab}(\mathbf{z}) + \frac{\alpha_s}{2\pi^2}\int_0^1 \frac{d\xi}{\xi_+(1-\xi)_+}\left(\xi^2+(1-\xi)^2+\xi^2(1-\xi)^2\right) \times \int\limits_{\mu^{-1}<|Z|<Q^{-1}} \frac{d^2Z}{Z^2}\left[D^{ab}(\mathbf{z}+(1-\xi)Z,\mathbf{z}-\xi Z) - N_c S^{ab}(\mathbf{z})\right] \quad (9)$$

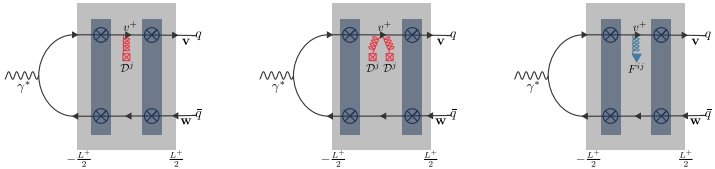
- HE evolution for dressed WL $\mathbb{S}^{ab}_Q$, which solve DGLAP (resumming $\ln X^2_\perp Q^2$) while $\ln X^2_\perp \mu^2_{\overline{MS}}$ is associated to r.c.

Part 2: Phenomenology using the CGC

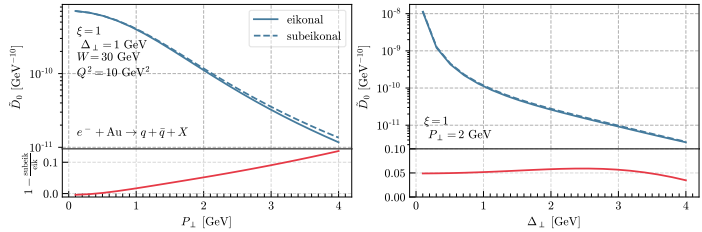
Many dijets: Dijet production at Next-to-eikonal

P. Agostini, T. Altinoluk and N. Armesto - JHEP 07 (2024) 137

- Dijet production in DIS
- Computation at next-to-eikonal, introduce decorated WLs. Contains field insertion beyond the large component of the gauge field.



- Beyond the shockwave approximation: splitting inside the shock, finite length.
- Evaluated in the dilute limit: WL expanded to order $\mathcal{O}(g^2)$, for both next-to-eikonal dipole and quadrupole.



- Dijet production in polarized DIS
- Computation at sub-eikonal accuracy using polarized WLs.
- Contains contribution of the $\gamma \to q\bar{q}$ splitting within the shock-wave (beyond sw approximation).
- In the back-to-back kinematic limit probes the Weizsäcker-Williams gluon helicity TMD.

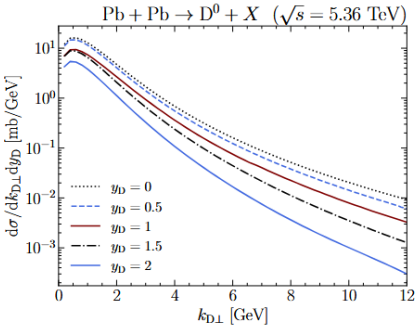
$$\sum_{\lambda=\pm 1} \lambda\, z(1-z)\, \frac{d\sigma^{\gamma^* p \to q\bar{q}X}_{\lambda\lambda}}{d^2p\, d^2\Delta\, dz} \approx -\frac{2\,\alpha_s\,\alpha_{EM}\,Z_f^2}{s} \left[z^2 + (1-z)^2\right] \frac{p_T^2 - a_f^2}{(p_T^2 + a_f^2)^2}\, g_{1L}^{G\,WW}\left(x \approx \frac{p_T^2}{s}, \Delta_T^2\right). \quad (10)$$

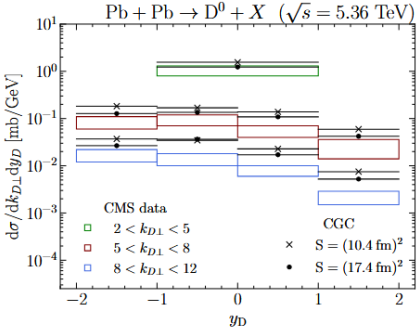
- Study the small-x evolution of this TMD at DLA in the large N_c-limit.
- Outside the saturation region, show that $g_{1L}^{G\,WW}(x, k_\perp^2) \sim g_{1L}^{G\,dip}(x, k_\perp^2)$. Related to the evolution of the same operator G_2.
- Can be used to further constrain the proton spin at low-x.

Meson production: Inclusive D^0 production in UPC

[P. Gimeno-Estivill, T. Lappi and H. Mantysaari - Phys.Rev.D 111 (2025) 11, 114036]

- Compute the cross section for $d\sigma^{A+A\to D^0+X}/dy_D d^2k_{D\perp}$ in UPCs
- Partonic process $\gamma + A_\mu^{cl} \to q\bar{q}$.
- Probe the dipole $D = \langle \mathrm{tr}\, V V^\dagger \rangle$ [antiquark integrated out, thus quadrupole collapses]
- Dipole evolved according to rcBK from an initial condition MV^e fitted to HERA-data
- Compared to CMS-data.

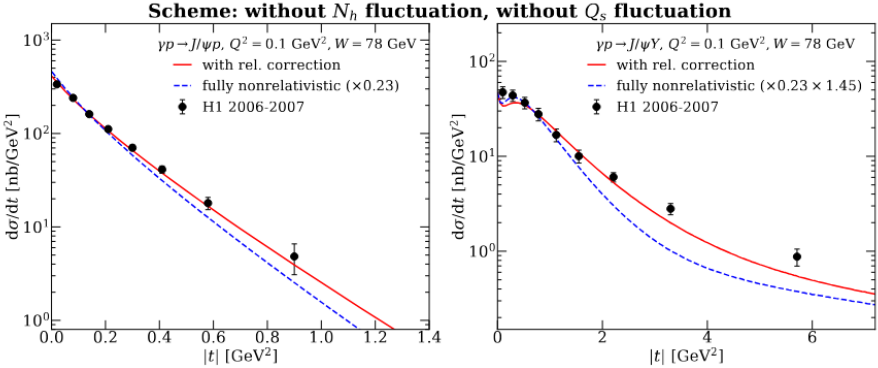




Meson production: Coherent vs incoherent in exclusive J/ψ production

[H. Mantysaari and A. Le - 2509.07480]

- Cross section for exclusive J/ψ production.
- Consider coherent and incoherent mechanisms. Probe either the average of configurations $|\langle \mathrm{tr}\, VV^\dagger \rangle|^2$ or the variance of the configurations $\langle |\mathrm{tr}\, VV^\dagger|^2 \rangle - |\langle \mathrm{tr}\, VV^\dagger \rangle|^2$.
- Hot spot model for the initial condition. No evolution, but can be implemented using JIMWLK or BK.



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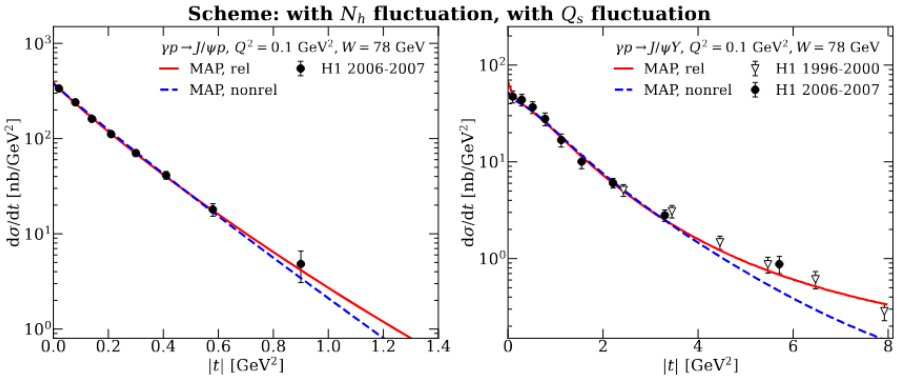
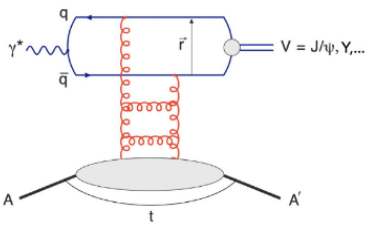


Exclusive heavy vector meson production: Gluon saturation at the LHC?

C. Royon and J. Penttala - Phys.Lett.B 864 (2025) 139394



- Heavy vector meson production in UPC at LHC. (proton or heavy ion target)
- γ-pomeron interaction: probe the low-x gluon distribution of the target.
- Gluon saturation? Probe with a low scale compared to Q_s, while perturbative.
- Low p_t for c or b production; or J/ψ production are ideal probes.

Calculation:

- BFKL or BK evolution.
- Impact parameter b-dependence in the dipole amplitude.
- Gaussian thickness for the proton / Wood-Saxon for heavy ion Pb.
- Adjusted BFKL to a fit of the vector meson data in the proton.

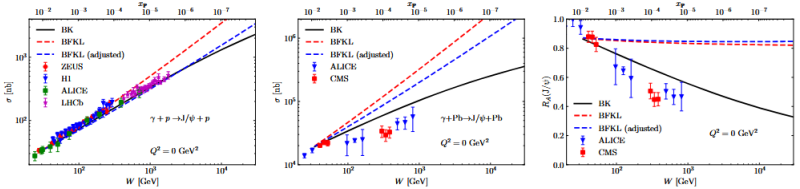


Figure 5: Exclusive $J\Psi$ production as a function of the center-of-mass energy W. Left: Proton target, Center: Lead target, Right: Nuclear suppression factor,

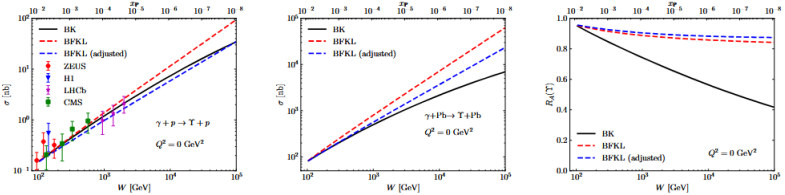


Figure 6: Exclusive Υ production as a function of the center-of-mass energy W. Left: Proton target, Center: Lead target. Right: Nuclear suppression factor.

Recap

Recap:

- Accuracy of CGC calculation has been improved by considering either NLO corrections or beyond eikonal approximation.
- Efforts are currently ongoing on unifying DGLAP and HE evolution
- Phenomenology for the EIC and LHC data is being considered by the community in a growing set of observables within the CGC framework.