Hierarchy depth in directed networks

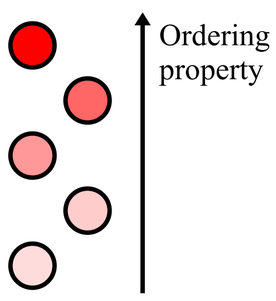
K. Suchecki, J.A. Hołyst

Center of Excellence for Complex Systems Research, Faculty of Physics
Warsaw University of Technology, Koszykowa 75, PL-00-662 Warsaw, Poland

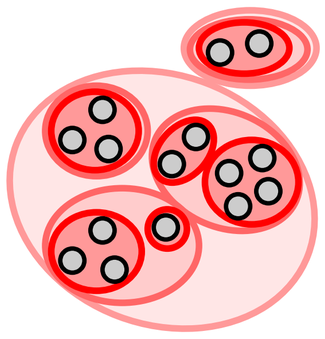
The term "hierarchy" when applied to networks can mean one of the few structures: simple order hierarchy meaning ordering of elements, nested hierarchy that is multi-level community structure or flow hierarchy defined by directed links that show causal or control structure in the network.
We introduce two measures of node depth for flow hierarchy in directed networks, measuring the level of hierarchy the nodes belong to. Rooted depth is defined as distance from specific root node and relative depth relies on directed links serving as "target node has larger depth than source" relations. We explore the behavior of these two measures, their properties and differences between them. To cope with eventual directed loops in the networks we introduce loop-collapse method, that evens out depth values for all nodes in the same directed loop. We investigate the behavior of the introduced depth measures in random graphs of different sizes and densities as well as some real network topologies. Maximum depth depends on network density, first increasing with mean degree, up to percolation threshold and declining afterwards as the number of loops increase.

Hierarchy

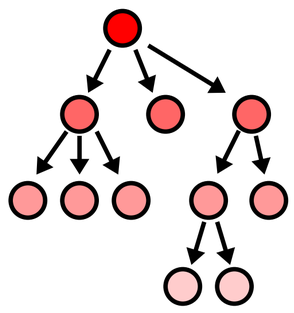
Order hierarchy
Ordering by some property



Nested hierarchy
Grouping at different scales



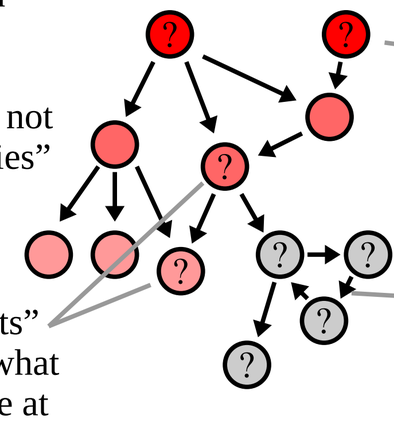
Flow hierarchy
Multi-level structure by relations



Different types of hierarchies exist in reality.
Research question: how to define depth or level of nodes in flow hierarchies ?

How to define it for networks that do not form perfect hierarchy ?

Problems with defining flow hierarchy for networks that are not "perfect hierarchies"



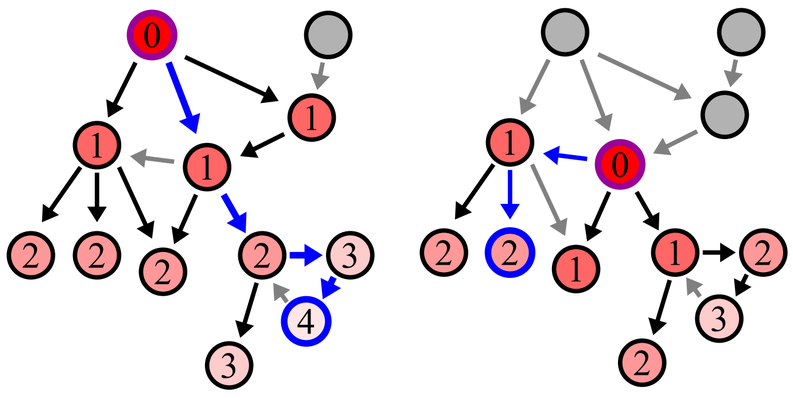
multiple "roots"
Impossible to determine the top in certain situations

multiple "roots"
Ambiguity in what level nodes are at

directed loops
No way to define hierarchy unambiguously

Definitions of depth

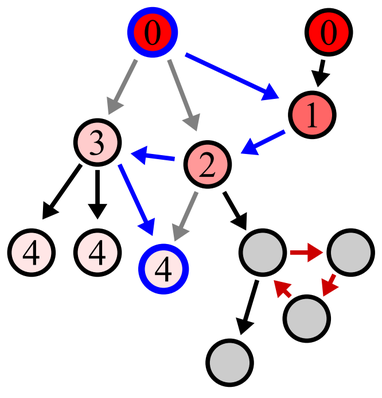
Rooted depth



Defined by: how far away from **root** it is.
Is equal to: length of **shortest** path from **root** to given node.

Issues:
- depends on choice of the root (different values)
- not all nodes have defined value for given root

Relative depth

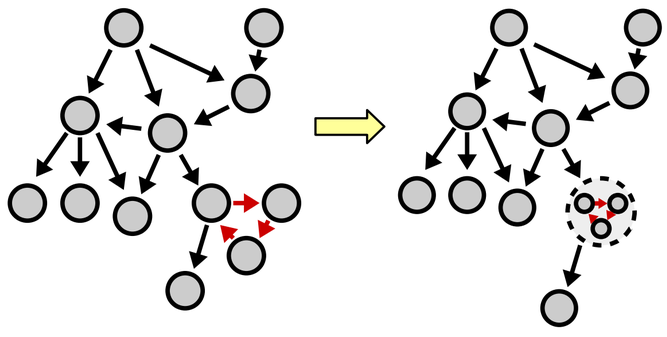


Defined by: **link** targets have higher depth than link sources (minimum 1)
Is equal to: length of **longest** path from any zero depth to given node

Issues:
- ambiguous zero depth position
- cannot handle directed cycles

How to deal with cycles/loops ?

Loop collapse

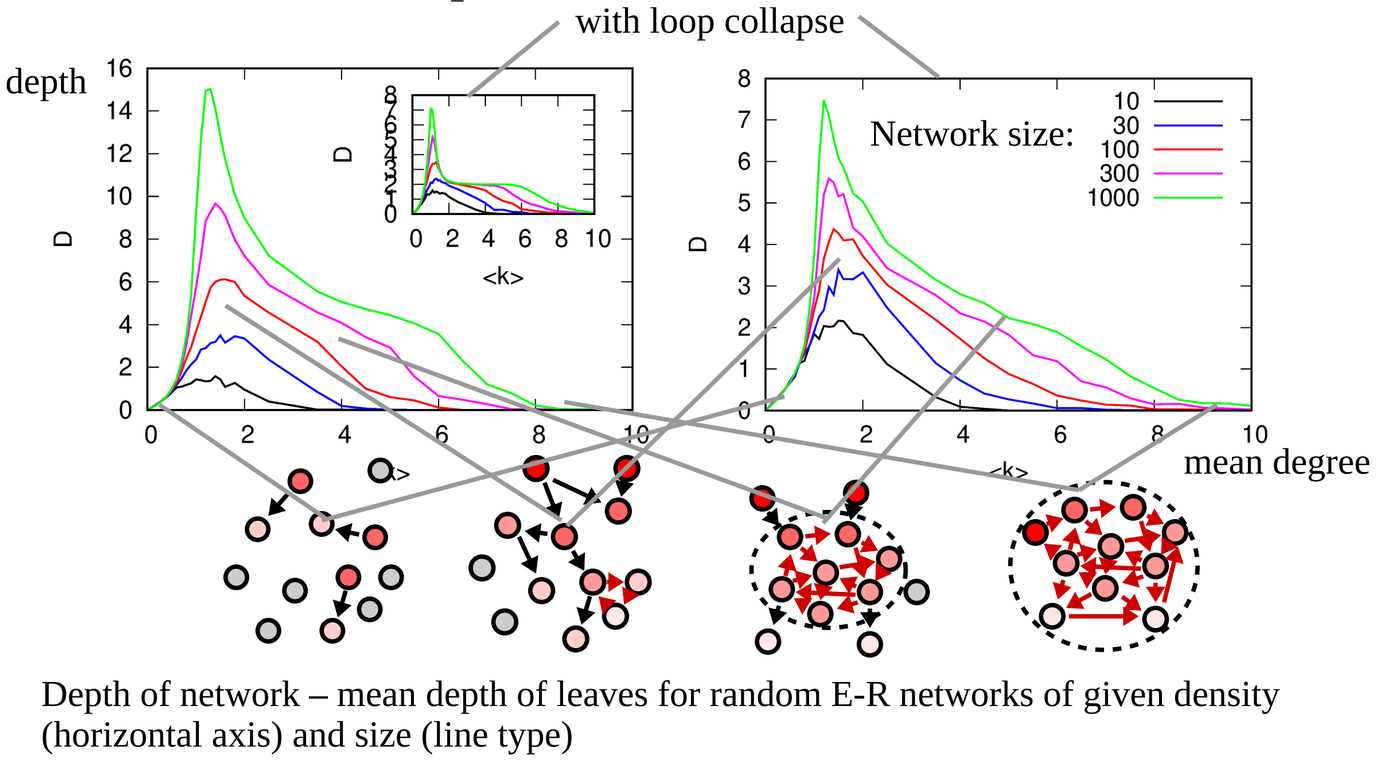


Treat all nodes of any directed cycle/loop as single node when determining depth (all will have same depth)

- required for relative depth
- optional for rooted depth

Results for random and real networks

Rooted depth



Depth of network – mean depth of leaves for random E-R networks of given density (horizontal axis) and size (line type)

$$D_{root} = \langle d_{leaf}^{(root)} \rangle$$

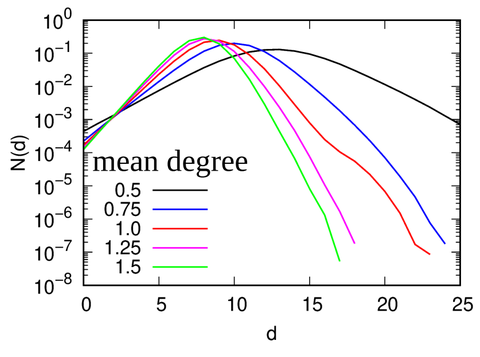
Relative depth

$$D_{rel} = \langle d_{leaf} \rangle - \langle d_{root} \rangle$$

Rooted depth – random vs real

Distributions of node depths

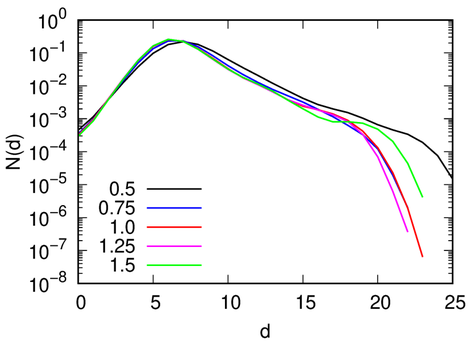
Random E-R network



Random graph – density plays decisive role

(percolation decides how the network looks like and resulting depths, see on the left)

Gnutella network 4.8.2002



Real network – structure exists independently from density

(network has fraction of links removed to result in specified mean degree)

results to be published: K. Suchecki, J.A. Hołyst, "Hierarchy depth in directed networks", in preparation