Generic Features in the Spectral Decomposition of Correlation Matrices

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Abstract

We show [1] that correlation matrices with particular average and variance of the correlation coefficients have a notably restricted spectral structure. Applying geometric methods, we derive lower bounds for the largest eigenvalue and the alignment of the corresponding eigenvector. We explain how and to which extent, a distinctly large eigenvalue and an approximately diagonal eigenvector generically occur for specific correlation matrices independently of the correlation matrix dimension.

Why correlation matrices?

Correlation matrices are widely used across scientists and practitioners, especially in finance. An $n \times n$ correlation matrix $C = M^T M$ is a product of an $N \times n$ matrix $M := [r_1, \ldots, r_n]$ and its transpose M^T. The columns of M are normalised $r_i^T r_i \equiv 1$. Hence, a real $n \times n$ correlation matrix $C = (C_{ij})$: is symmetric, it has ones on the the diagonal, and it is positive semi-definite. Usually the matrix M contains n time series of length N as columns. Here we consider C as a fixed realisation of a random variable and we do not refer to any matrix M.

Spectral Decomposition vs. Characteristic of Correlation Matrices

Characteristic: For every correlation matrix C we consider

the mean $\quad c := \dfrac{2}{n(n-1)} \displaystyle\sum_{i>j} C_{ij} \quad$ and the standard deviation $\quad \sigma := \sqrt{\dfrac{2}{n(n-1)} \displaystyle\sum_{i>j} C_{ij}^2 - c^2},$

of the coefficients C_{ij}. We denote the mapping $C \mapsto (c, \sigma)$ as *the characteristic* of C. The characteristic is always a point in the upper half of the unit disk in the (c, σ)-plane. Furthermore, we found that for avery $n \times n$ correlation matrix C one always has $c \geq 1/(1-n)$.

Spectral Decomposition: Every $n \times n$ correlation matrix C has a spectral decomposition

$$C = \sum_{i=1}^{n} \lambda_i v_i v_i^T,$$

with the real eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ and an orthonormal basis $v_1, .., v_n \in \mathbb{R}^n$ of the corresponding eigenvectors. We introduce the normalised diagonal vector $\delta_n := (1, \ldots, 1)/\sqrt{n} \in \mathbb{R}^n$, and the weights

$$w_j := \langle v_j, \delta_n \rangle^2.$$

The weights measure the angle between the diagonal vector δ_n and the eigenvectors. We note that the largest weight

$$w_{\max} := \max_{n \geq j \geq 1} (w_j),$$

corresponds to the most diagonal eigenvector. For the weights and the eigenvalues ones has

$$\sum_{j=1}^{n} \lambda_j / n = \sum_{j=1}^{n} w_j = 1 \quad \text{and} \quad 0 \leq w_j, \lambda_j/n \leq 1.$$

Therefore the quantities $\lambda_1/n, w_1, w_{\max} \in [0, 1]$ can be compared for C of any dimension n.

References

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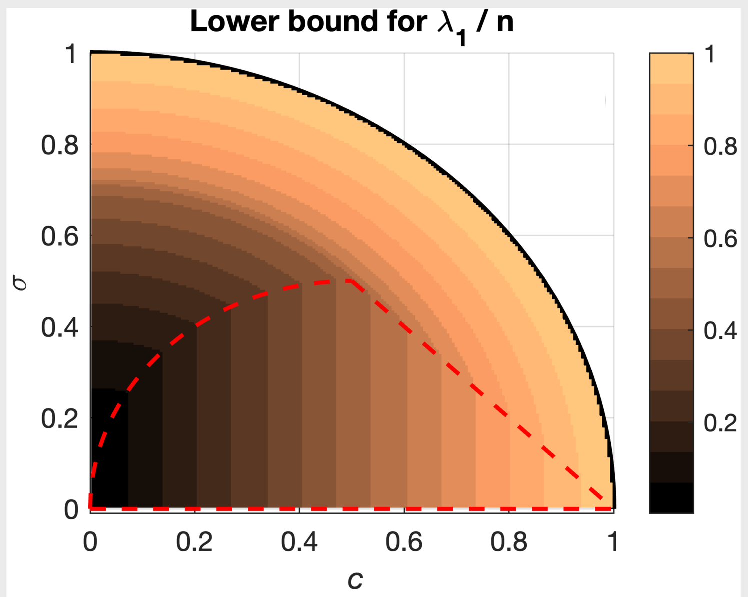
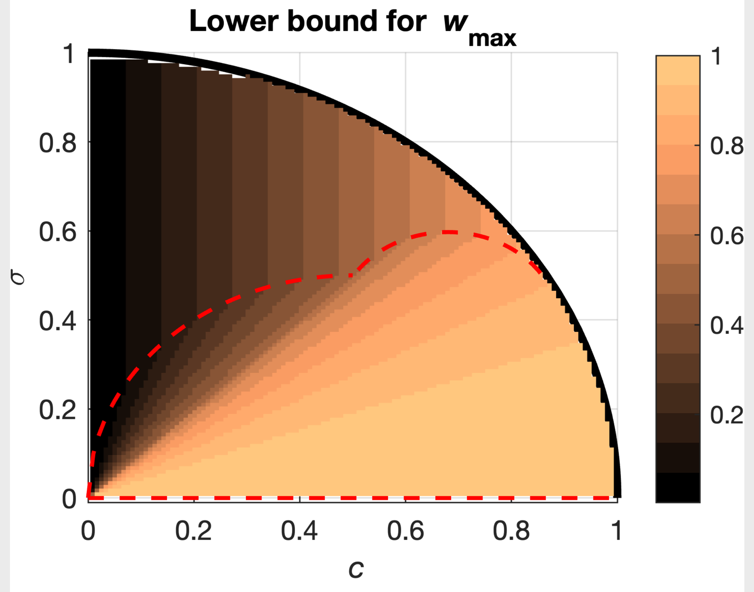
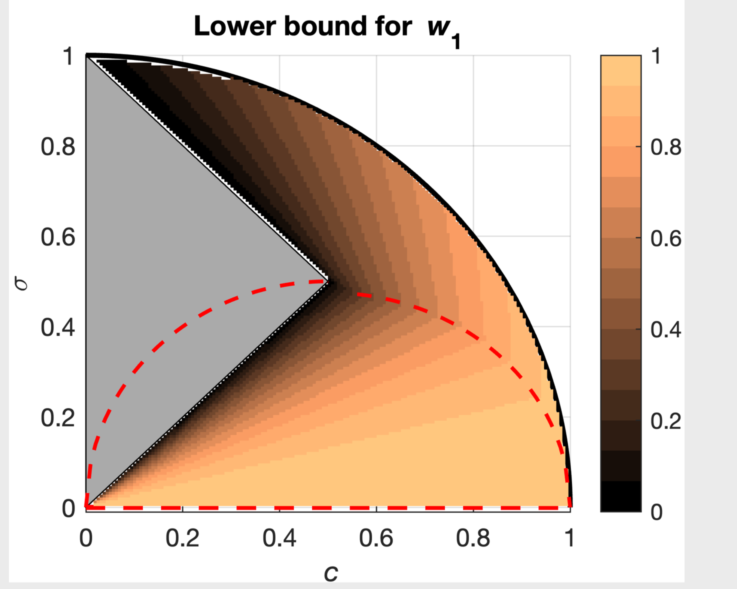
Main Results: Relation Between (c, σ) and λ_1, w_1, $w_{\max}$.

Universal Bounds: Consider the function s defined by $s(x) := \frac{1}{2}\left(1 + \sqrt{2x-1}\right)$ if $x \geq 1/2$, and $s(x) := x$ if $0 \leq x < 1/2$. For every $n \times n$ correlation matrix $C \neq \mathrm{Id}$ we arrive at the following estimates

$\lambda_1/n \geq \max\{c, s(c^2+\sigma^2)\}$ $\qquad$ $w_{\max} \geq \max\left\{c, s\left(c^2/(c^2+\sigma^2)\right)\right\}$ $\qquad$ If $c > 0$ then $w_1 \geq 1 - \min\left\{\sigma^2/c^2, (1-c)/s(c^2+\sigma^2)\right\}$





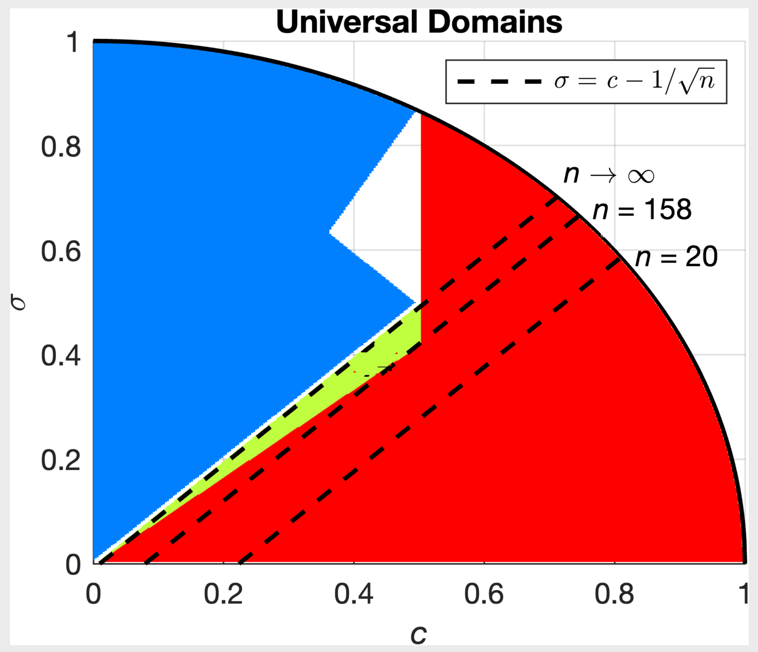


We extend the results of Refs. [2,3,4] for correlation matrices. We find that a correlation matrix C has a distinctly large eigenvalue not only for $c \approx 1$ (as was shown by Meyer [4]), but more general for $c^2 + \sigma^2 \approx 1$. Furthermore, we find that an eigenvector of the largest eigenvalue is approximately diagonal not only when σ is small (as shown in Refs. [2,3]), but also when $c^2/(c^2+\sigma^2) > 4/5$. But for which correlation matrices holds $w_1 = w_{\max}$, *i.e.* when does the first eigenvector has the smallest angle with the diagonal?

Universal Domains: Empirical financial correlation matrices have $w_1 = w_{\max}$ [2]. We show that if the characteristic (c, σ) of an $n \times n$ correlation matrix C satisfies at least one of the conditions

(i) $c \geq 1/2$, (red)
(ii) $c \geq \sigma + 1/\sqrt{n}$, (green or dashed)
(iii) $c \geq \sqrt[3]{2}\sigma$, (red)

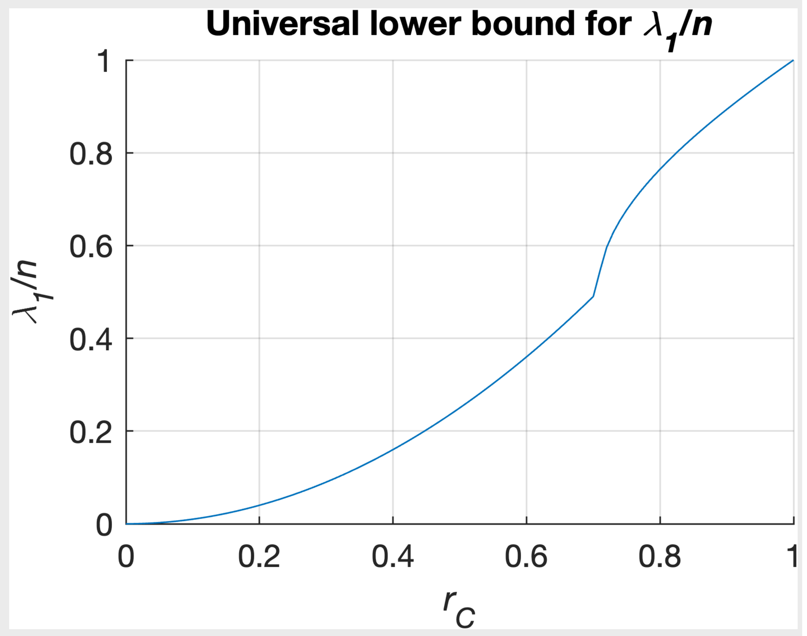
then one has $w_1 > \frac{1}{2}$ and hence $w_1 = w_{\max}$. For any (c, σ) in the blue domain, and any sufficiently large n, we find correlation matrices with $w_1 < w_{\max}$.



Polar coordinates: Let $\Theta \in [0, \pi]$ be the angle between δ_n and an eigenvector v_1 for the largest eigenvalue λ_1. We rewrite the characteristic $(c, \sigma) = (r_C \cos(\phi_C), r_C \sin(\phi_C))$ in polar coordinates. It follows that for every $n \times n$ correlation matrix C with $w_1 = w_{\max}$ one has

$$\lambda_1/n \geq s(r_C^2) \quad \text{and} \quad \Theta \leq \phi_C.$$

Surprisingly, ϕ_C in the (c, σ)-plane bounds Θ in $\mathbb{R}^n$.



Methodology

Characteristic Lemma: We define the scaling function $g_n(x) := ((n-1)x + 1)/n$. We find that for an $n \times n$ ($n \geq 2$) correlation matrix C, with the mean correlation c and the standard deviation σ, the eigenvalues $\lambda_1, \ldots, \lambda_n$ and an corresponding eigenbasis $v_1, \ldots, v_n$, one has

$$\langle \tilde{\lambda}, w \rangle = g_n(c) \qquad \text{and} \qquad \|\tilde{\lambda}\|^2 = g_n(c^2 + \sigma^2).$$

Here $\tilde{\lambda} := (\lambda_1, \ldots, \lambda_n)/n$ denotes the normalised eigenvalues vector and $w := (w_1, \ldots, w_n)$ denotes the weights vector, respectively.
Convexity: The space of $n \times n$ correlation matrices is convex. In particular, given a correlation matrix with characteristic (c, σ) we have that $C_\mu := \mu C + (1-\mu)\mathrm{Id}$ is a correlation matrix with characteristic $(c_\mu, \sigma_\mu) = \mu(c, \sigma)$ for any $0 \leq \mu \leq 1$.
Tensor Product: The tensor product of two correlation matrices is again a correlation matrix. The quantities $g_n(c)$ and $g_n(c^2 + \sigma^2)$ behave multiplicative under tensor products.

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