Generic Features in the Spectral Decomposition of Correlation Matrices

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Abstract

We show [1] that correlation matrices with particular average and variance of the correlation coefficients have a notably restricted spectral structure. Applying geometric methods, we derive lower bounds for the largest eigenvalue and the alignment of the corresponding eigenvector. We explain how and to which extent, a distinctly large eigenvalue and an approximately diagonal eigenvector generically occur for specific correlation matrices independently of the correlation matrix dimension.

Why correlation matrices?

Correlation matrices are widely used across scientists and practitioners, especially in finance. An correlation matrix $C$ is a matrix independently of the correlation matrix dimension.

Spectral Decomposition vs. Characteristic of Correlation Matrices

Characteristic: For every correlation matrix $C$ we consider

\[ \mu := \frac{1}{n(n-1)} \sum_{i<j} C_{ij} \]

and the standard deviation

\[ \sigma := \sqrt{\frac{1}{n(n-1)} \sum_{i<j} (C_{ij} - \mu)^2} \]

of the coefficients $C_{ij}$. We denote the mapping $C \mapsto (\mu, \sigma)$ as the characteristic of $C$. The characteristic is always a point in the upper half of the unit disk in the $(\mu, \sigma)$-plane. Furthermore, we found that for every $n \times n$ correlation matrix $C$ one always has $\mu \geq 1/\sqrt{n(n-1)}$.

Spectral Decomposition: Every $n \times n$ correlation matrix $C$ has a spectral decomposition

\[ C = \sum_{i=1}^{\min(n,1)} \lambda_i v_i v_i^T \]

with the real eigenvalues $\lambda_i \geq 0$ and an orthonormal basis $v_1, \ldots, v_n \in \mathbb{R}^n$ of the corresponding eigenvectors. We introduce the normalised diagonal vector \( \delta_i := (1, \ldots, 1)/\sqrt{n} \in \mathbb{R}^n \), and the weights $w_i := \langle v_i, \delta_i \rangle^2$.

The weights measure the angle between the diagonal vector $\delta_i$ and the eigenvectors. We note that the largest weight

\[ w_{\text{max}} := \max_{1 \leq i \leq n} (w_i) \]

corresponds to the most diagonal eigenvector. For the weights and the eigenvalues one has

\[ \sum_{i=1}^{\min(n,1)} \lambda_i / n = \sum_{i=1}^{\min(n,1)} w_i = 1 \quad \text{and} \quad 0 \leq w_i, \lambda_i / n \leq 1 \]

Therefore the quantities $\lambda_i / n, w_i, w_{\text{max}} \in [0,1]$ can be computed for $C$ of any dimension $n$.

Main Results: Relation Between $(c, \sigma)$ and $\lambda_1, w_1, w_{\text{max}}$.

Universal Bounds: Consider the function $s$ defined by $s(x) := \frac{1}{2} (1 + \sqrt{1 + 4x})$ if $x \geq 1/2$, and $s(x) := x$ if $0 \leq x \leq 1/2$. For every $n \times n$ correlation matrix $C \neq \mathbb{I} d$ we arrive at the following estimates

\[ \lambda_1 / n \geq \max \{ c, s(c^2 + \sigma^2) \} \]

\[ w_{\text{max}} \geq \max \{ c, s(c^2 + \sigma^2) \} \]

\[ w_1 \geq \min \{ \sigma^2 / c^2, (1 - c) / s(c^2 + \sigma^2) \} \]

We extend the results of Refs. [2,3,4] for correlation matrices. We find that a correlation matrix $C$ has a distinctly large eigenvalue not only for $c \approx 1$ as was shown by Meyer [4], but more general for $c^2 + \sigma^2 \approx 1$. Furthermore, we find that an eigenvector of the largest eigenvalue is approximately diagonal not only when $\sigma$ is small (as shown in Refs. [2,3]), but also when $c^2 / (c^2 + \sigma^2) > 4/5$. But for which correlation matrices holds $w_1 = w_{\text{max}}$, i.e. when does the first eigenvector has the smallest angle with the diagonal?

Universal Domains: Empirical financial correlation matrices have $w_1 = w_{\text{max}}$.

\[ \frac{\lambda_1}{n} \geq \max \{ c, \sigma \} \]

\[ w_{\text{max}} \geq \max \{ c, \sigma \} \]

If $c > 0$ then $w_1 \geq 1 - \min \{ \sigma^2 / c^2, (1 - c) / s(c^2 + \sigma^2) \}$

Polar coordinates: Let $\Theta \in [0, \pi]$ be the angle between $\delta_1$ and an eigenvector $v_1$ for the largest eigenvalue $\lambda_1$. We rewrite the characteristic $(c, \sigma) = (r \cos(\phi), r \sin(\phi))$ in polar coordinates. It follows that for every $n \times n$ correlation matrix $C$ with $w_1 = w_{\text{max}}$ one has

\[ \frac{\lambda_1}{n} \geq \sigma (c^2) \quad \text{and} \quad \Theta \leq \phi_0. \]

Surprisingly, $\phi_0$ in the $(c, \sigma)$-plane bounds $\Theta$ in $\mathbb{R}^2$.

Methodology

Characteristic Lemma: We define the scaling function $g_0(x) := (n-1)x + 1)/n$. We find that for an $n \times n$ (n $\geq 2$) correlation matrix $C$, with the mean correlation $c$ and the standard deviation $\sigma$, the eigenvalues $A_1, \ldots, A_n$ and an corresponding eigenvector $v_1, \ldots, v_n$ one has

\[ \langle \lambda_i, w_i \rangle = g_0(c) \quad \text{and} \quad | \lambda_i |^2 = g_0(c^2 + \sigma^2) \]

Here $\lambda := (A_1, \ldots, A_n)/n$ denotes the normalised eigenvalues vector and $w := (w_1, \ldots, w_n)$ denotes the weights vector, respectively.

Convexity: The space of $n \times n$ correlation matrices is convex. In particular, given a correlation matrix with characteristic $(c, \sigma)$ we have that $C_{\mu} := \mu C + (1 - \mu) \mathbb{I}$ is a correlation matrix with characteristic $(c_{\mu}, \sigma_{\mu}) = (\mu, \sigma)$ for any $0 \leq \mu \leq 1$.

Tensor Product: The tensor product of two correlation matrices is again a correlation matrix. The quantities $g_0(c)$ and $g_0(c^2 + \sigma^2)$ behave multiplicatively under tensor products.

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\[ \text{References} \]


\[ \text{Figure 1: Illustration of the spectral decomposition of a correlation matrix.} \]

\[ \text{Figure 2: Plot of the lower bound for } \lambda_1 / n \text{ and } w_{\text{max}}. \]

\[ \text{Figure 3: Polar plot of the characteristic function.} \]